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Fractional Ornstein-Uhlenbeck Processes

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Abstract

In this monograph, we are mainly studying Gaussian processes, in particularly three different types of fractional Ornstein–Uhlenbeck processes. Pioneers in this field may be mentioned, e.g. Kolmogorov (1903-1987) and Mandelbrot (1924-2010).

The Ornstein–Uhlenbeck diffusion can be constructed from Brownian motion via a Doob transformation and also from a solution of the Langevin stochastic differential equation. Both of these processes have the same finite dimensional distributions. However the solution of the Langevin stochastic differential equation, which driving process is fractional Brownian motion and a Doob transformation of fractional Brownian motion do not have same finite dimensional distributions. Indeed we verify, that the covariance of the fractional Ornstein–Uhlenbeck process of the first kind (which we call the solution of the Langevin stochastic differential equation in which the driving process is fractional Brownian motion) behaves at infinity like a power function and the covariance of the fractional Ornstein–Uhlenbeck process (constructed by a Doob transformation of fractional Brownian motion) behaves at infinity like an exponential function. Moreover we study the behaviour of the covariances of these fractional Ornstein–Uhlenbeck processes. We also calculate the spectral density function for the Doob transformation of fractional Brownian motion using a Bochner theorem.

We present the Doob transformation of fractional Brownian motion via solution of the Langevin stochastic differential equation. One of the main aims of our research is to analyse its driving process. This driving process is $Y_t^{(\alpha)} = \mathrm{e}^{-t\alpha} Z_{\tau_t}$, where $\tau_t = \frac{He^{\frac{\alpha t}{H}}}{\alpha}$ and $\{Z_t : t \geq 0\}$ is fractional Brownian motion. We find out that the process $Y^{(\alpha)} := \{Y_t^{(\alpha)} : t \geq 0\}$, if scaled properly, has the same finite dimensional distributions as the process $Y^{(1)} := \{Y_t^{(1)} : t \geq 0\}$. The main result in this monograph is that we define a stationary fractional Ornstein–Uhlenbeck process of the second kind as a process with a two-sided driving process $\{\widehat{Y}_t^{(1)} : t \in \mathbb{R}\}$ and create a new family of fractional Ornstein-Uhlenbeck processes. We study many properties of the fractional Ornstein–Uhlenbeck process of the second kind. For example, we show that the fractional Ornstein–Uhlenbeck process of the second kind is Hölder continuous of any order $\beta < H$ and find the kernel representation of its covariance.

We research many properties of the processes $Y^{(\alpha)}$ and $Y^{(1)}$, since they are quite interesting themselves. We represent these processes as stochastic integrals with respect to Brownian motion and prove that the sample paths of the process $Y^{(\alpha)}$ are Hölder continuous of any order $\beta < H$. In the case $H \in (\frac{1}{2}, 1)$, we find out the covariance kernel of increment process of $Y^{(\alpha)}$, and using that we investicate the covariance of $Y^{(\alpha)}$ and the variance of $Y^{(\alpha)}$, when t tends to infinity. One of our main results is that the increment process of $Y^{(\alpha)}$ is short-range dependent. We also study weak convergence and tightness and then finally prove that $\frac{1}{\sqrt{a}} Y_{at}^{(\alpha)}$ converges weakly to scaled Brownian motion.

In the case $H \in (\frac{1}{2}, 1)$, fractional Brownian motion and the fractional Ornstein – Uhlenbeck process of the first kind both exhibit a long-range dependence, but the fractional Ornstein – Uhlenbeck process of the second kind exhibits a short-range dependence. This offers more opportunities to model network traffic or economic time series via tractable fractional processes. The fractional Ornstein – Uhlenbeck process of the first kind and the fractional Ornstein – Uhlenbeck process of the second kind are quite similar to simulate, since they can both be represented via stochastic differential equations.

Preface

The work presented in this thesis and the post-graduate studies including the Licentiate thesis were carried out in the Department of Mathematics at the University of Joensuu and the Department of Mathematics at Tampere University of Technology.

I want to express my deepest thanks to my supervisors Prof. Paavo Salminen and Prof. Sirkka-Liisa Eriksson for their skilful guidance and encouragement during my doctoral studies. I also want thank all people from the Finnish Graduate School in Stochastics: Paavo, Göran, Ilkka et al. not forgetting late Esko, you have made the best place ever to study stochastics, the atmosphere is unique! I would like to thank pre-examiners Professor Tommi Sottinen and Adjunct Professor Ehsan Azmoodeh for their encouraging comments and careful reading. In addition, I would like to thank Professor Ilkka Norros, who wanted to be the opponent in the public defence of this monograph. I thank Heikki Orelma and Osmo Kaleva for discussions and Osmo and Simo Ali-Löytty for guidance with Matlab and LaTeX. I am also grateful to all my colleagues in the University of Joensuu and Tampere University of Technology. I want also thank Virginia Mattila and Anu Granroth, who also helped me with the language and grammar.

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Tampere, September 2015

Terhi Kaarakka

Contents

Mathematical notations

List of notations

$\mathcal{B}(U)$	Borel σ-algebra of subsets of U
$\mathcal{F}_t$	Filtration
$\mathbb{N}$	Natural numbers, $\{1, 2, 3, \ldots\}$
$\mathbb{N}_0$	Natural numbers $\cup \{0\}$, $\{0, 1, 2, 3, \ldots\}$
$\mathbb{R}$	Real numbers
$H \in (0, 1)$	Constant of Hurst
$\binom{2H}{n} = \frac{2H(2H-1)\cdots(2H-n+1)}{n!}$	Binomial coefficient, $n > 0$, $H \in (0, 1)$
$\tau_t = \frac{H\mathrm{e}^{\frac{\alpha}{H}t}}{\alpha}$	
$\tau_t^{(1)} = H\mathrm{e}^{\frac{t}{H}}$	
Δ	Odd increasing density function
Δ'	Spectral density function
Γ	Gamma function
$\mathbf{E}(X) = \mu$	Mean
$\mathbf{E}\left((X_t - \mu_t)(X_s - \mu_s)\right) = \mathrm{cov}(X_t, X_s)$	Covariance
F	Distribution function
$\mathbf{P}(X_s < y)$	Distribution of X_s is less than y
$\mathbf{P}(X_s < y \mid \mathcal{F}_t)$	Conditional distribution of X_s given $\mathcal{F}_t$
$\rho_X(n) = \mathbf{E}(X_i X_{i+n})$	Covariance, when mean is zero and i is arbitrary non-negative integer
$\mathbf{Q}(s, t) = \mathbf{Q}(t - s) = \mathbf{E}(X_t X_s)$	Covariance, when mean is zero
$\wedge : \widehat{f}(\gamma) = \int f(x)\mathrm{e}^{i\gamma x} dx$	Fourier transformation of a function f
$\vee : \check{f}(x) = \frac{1}{2\pi} \int f(\gamma)\mathrm{e}^{-i\gamma x} d\gamma$	Inverse Fourier transformation of a function f

$\{X_t : t \in \mathbb{R}\}$	Stochastic process, t is time
$\{B_t : t \geq 0\}$	Brownian motion, Bm
$\{\widehat{B}_t : t \in \mathbb{R}\}$	Two-sided Brownian motion
$\{Z_t : t \geq 0\}$	Fractional Brownian motion, fBm
$\{\widehat{Z}_t : t \in \mathbb{R}\}$	Two-sided fractional Brownian motion
$I_Z = \{Z_n - Z_{n-1} : n = 0, 1, 2, \ldots\}$	Increment process of fBm
$\{V_t : t \in \mathbb{R}\}$ or $\{U_t : t \in \mathbb{R}\}$	Ornstein−Uhlenbeck process, OU
$\{U_t^{(Z,\alpha)} : t \in \mathbb{R}\}$	Fractional Ornstein−Uhlenbeck process of the first kind, fOU(1)
$\{X_t^{(D,\alpha)} : t \in \mathbb{R}\}$	Fractional Ornstein−Uhlenbeck process or Doob transformation of fBm, fOU
$\{U_t^{(D,\gamma)} : t \in \mathbb{R}\}$	Fractional Ornstein−Uhlenbeck process of the second kind, fOU(2)
$\{Y_t^{(\alpha)} : t \in \mathbb{R}\}$	$Y_t^{(\alpha)} = \int\limits_0^t \mathrm{e}^{-\alpha s} dZ_{\tau_s}$
$I_Y = \{Y_t^{(\alpha)} - Y_s^{(\alpha)} : s, t \in \mathbb{R}\}$	Increment process of $\{Y^{(\alpha)}\}$
$\{\widehat{Y}_t^{(1)} : t \in \mathbb{R}\}$	Two-sided driving process of fOU(2)
$k_X(u, v)$	Kernel in the representation of the covariance of process $\{X_t : t \geq 0\}$
$k_{\alpha H}(u - v) = r_{\alpha H}(u, v)$	Kernel in the representation of the covariance of $\{Y^{(\alpha)_t : t \geq 0}\}$
$\{X_t : t \geq 0\} \stackrel{d}{=} \{Y_t : t \geq 0\}$	Processes $\{X_t : t \geq 0\}$ and $\{Y_t : t \geq 0\}$ have the same finite dimensional distribution
$P_n \stackrel{w}{\Rightarrow} P$	P_n converges weakly to P.
$\{X_t : t \geq 0\} \stackrel{d}{=} \{Y_s : s \geq 0\}$	$\{X_t : t \geq 0\}$ converges weakly to $\{Y_s : s \geq 0\}$ in the space of continuous functions.

1 Introduction

The main topics of this dissertation in the field of stochastic are fractional Ornstein[1]−Uhlenbeck[2] processes, that are special types of Gaussian[3] processes. This area has been studied by several authors. There are many publications devoted to fractional Ornstein−Uhlenbeck processes, e.g., Klepsyna and Le Breton [33], Cheridito [11], Mashui and Shieh [40], Nualart and Hu [48] and Azmoodeh and Viitasaari [3] or Morlanes [2]. We also find out that the Wolfram Demonstration Project has a demonstration of the fractional Ornstein−Uhlenbeck process [37]. In this software, there is an interactive simulation with the options to choose from, for example, the mean, the variance and the Hurst constant H.

It is well-known that the Ornstein−Uhlenbeck diffusion can be constructed from Brownian motion via a Doob[4] transformation as well as a solution of the Langevin [5] stochastic differential equation (see Doob [16]). Both of these processes have the same finite dimensional distributions. We thought that this would be the same for the fractional Ornstein−Uhlenbeck processes, but noticed fairly soon that this is not the case, since the covariance of the fractional Ornstein−Uhlenbeck process as a solution of the Langevin stochastic differential equation (abbreviated fOU(1)) behaves at infinity like a power function and the covariance of the fractional Ornstein−Uhlenbeck process constructed by the Doob transformation of fractional Brownian motion (abbreviated fOU) behaves at infinity like an exponential function. (A detailed discussion is presented in Chapter 2)

We present the Doob transform of fractional Brownian motion via the Langevin stochastic differential equation. One of the main objects is to analyse the driving process of this stochastic differential equation. The driving process of the Langevin equation is $Y_s^{(\alpha)} = \mathrm{e}^{-s\alpha} Z_{\tau_s}$, where $\tau_s = \frac{He^{\frac{\alpha s}{H}}}{\alpha}$ and $\{Z_t : t \geq 0\}$ is fractional Brownian motion (abbreviated fBm). We find out that the process $Y^{(\alpha)}$, if scaled properly, has the same finite dimensional distributions as the process $Y^{(1)}$. We define a stationary fractional Ornstein−Uhlenbeck process of the second kind (abbreviated fOU(2)) as a process in which a driving process is the two-sided process $\{\widehat{Y}_t^{(1)} : t \in \mathbb{R}\}$ (see Definition 3.6)

$$U_t^{(D,\gamma)} = \mathrm{e}^{-\gamma t} \int\limits_{-\infty}^{t} \mathrm{e}^{\gamma s} d\widehat{Y}_s^{(1)} = \mathrm{e}^{-\gamma t} \int\limits_{-\infty}^{t} \mathrm{e}^{(\gamma-1)s} dZ_{\tau_s^{(1)}},\ \gamma > 0, \tag{1.1}$$

[1]Leonard Ornstein (1880-1941), Dutch physicist.
[2]George Uhlenbeck (1900-1988), Ducht physicist.
[3]Karl F. Gauss (1777-1855), German mathematician.
[4]Joseph L. Doob (1910-2004), American mathematician.
[5]Paul Langevin (1872-1946), French physicist.

where $\tau_s^{(1)} = He^{\frac{s}{H}}$, $H \in (0,1)$ and Z is fractional Brownian motion. This fOU(2) coincides with fOU (the Doob transformation of fBm), when $\gamma = 1$.

One major motivation for studying these fractional Ornstein – Uhlenbeck processes is that if $H > \frac{1}{2}$, fractional Brownian motion and the fractional Ornstein – Uhlenbeck process of the first kind both exhibit a long-range dependence, but the fractional Ornstein – Uhlenbeck process of the second kind exhibits a short-range dependence. This offers more options to model network traffic or economic time series via tractable fractional processes. However, the fractional Ornstein – Uhlenbeck process of the first kind and the fractional Ornstein – Uhlenbeck process of the second kind are quite similar to simulate, since they can all be represented via stochastic differential equations. In Subsection 3.1.2 of Section 3.1 we present some simulations of these processes.

The fractional Ornstein – Uhlenbeck process of the second kind, $U^{(D,\gamma)}$, can be defined via a stochastic differential equation, where the driving process is the two-sided process $\widehat{Y}^{(1)}$. The process $\{Y_t^{(1)} : t \geq 0\}$ itself is interesting, since it is also a similar type of stochastic process as another fOU and

$$Y_t^{(\alpha)} = \int_0^t \mathrm{e}^{-\alpha s} dZ_{\frac{He^{\frac{\alpha s}{H}}}{\alpha}},$$

scaled properly, has the same finite dimensional distributions as the process $Y^{(1)}$ (in Chapter 3). We also study properties of weak convergence and tightness and then prove that $\frac{1}{\sqrt{a}} Y_{at}^{(\alpha)}$ converges weakly to the scaled Brownian motion.

We also study other properties of $U^{(D,\gamma)}$ and $Y^{(\alpha)}$. For example, we verify that they are locally Hölder continuous of the order $\beta < H$, $Y^{(\alpha)}$ has stationary increments and $U^{(D,\gamma)}$ is stationary. We find the kernel representation of the covariance of the increment process of $Y^{(\alpha)}$ and the process $U^{(D,\gamma)}$ and using these representations we find many other properties of these processes. One of the main results is that the both processes $U^{(D,\gamma)}$ and the increment process of $Y^{(\alpha)}$ are short-range dependent.

In order to make this monograph reader-friendly, we recall in Chapter 1 the basic definitions and properties of the Gaussian processes. We also recall stationarity and self-similarity and define some important Gaussian processes: Brownian motion, two Ornstein – Uhlenbeck processes, fractional Brownian motion and the fractional Ornstein – Uhlenbeck process as a solution of the Langevin stochastic differential equation and the fractional Ornstein – Uhlenbeck process as the Doob transformation of fractional Brownian motion. We present and prove numerous properties of these processes.

We calculate the spectral density function for the Doob transformation of fractional Brownian motion, using a Bochner theorem. To make the representation self-contained, the Bochner theorem is also given. We recall that in the Bochner theorem the covariance of the process is expressed as an integral with respect to its spectral density function (in Chapter 2, Sections 2.3 and 2.4).

Collecting everything together, our aim is to write a clear self-explanatory monograph, dealing with different fOU processes and their important properties.

In mathematics it is a habit to write things using "we" form, since we think that in a process of understanding there is a writer and a reader together. This means that the personal pronoun we is actually me and a reader.

The brief list of the novelty values and author's role in all chapters of the monograph are the following:
1. Introduction. In this chapter I compile the theoretical background, I recall basic definitions and theorems. There are plenty of proofs of mine, but only with minor novelty value.
2. Covariance and spectral density functions. I have written this chapter using my licentiate theses [28], which is written in Finnish. I have also developed further its ideas and improved results. Some theorems, for example, Corollary 2.4 is new. The main theorem of this chapter is Theorem 2.8, where the spectral density function of the Doob transformation of fBm is given, this theorem has also a novelty value.
3. Fractional Ornstein-Uhlenbeck processes. All results in this chapter are novelties. Fractional Ornstein-Uhlenbeck processes of second kind is defined for the first time in the publication [29], where I was the corresponding author and did the main mathematical work. In this chapter I have written propositions with complete proofs and some of them were already published (there is references in title) in [29], but with brief proofs. The publication [29] was important to publish fast, since results have strong novelty value.
4. Weak convergence. In Section 4.1., I recall the main concept of the weak convergence and in Section 4.2., I show that the driving process of fOU(2), $Y^{(\alpha)}$, if scaled properly, converges weakly to scaled Brownian motion. This novelty result and proof of mine is also published in [29].

1.1 Gaussian processes

1.1.1 Basic properties of Gaussian processes

In this section we recall some important definitions and properties that we use the most in this dissertation. They are quite standard in the literature. There are several good references on this subject. We mention, for example, Doob [17] and Dym and McKean [18].

Definition 1.1. A real-valued stochastic process $\{X_t : t \in \mathbb{R}\}$ in the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is called *a Gaussian process* if the vector

$$(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$$

is multivariate Gaussian for every $t_1, t_2, \ldots, t_n \in \mathbb{R}$, $n \geq 1$, i.e., every finite collection of random variables has a multivariate normal distribution.

It is well-known that the distribution of a Gaussian process $\{X_t : t \in \mathbb{R}\}$ is determined uniquely by its mean function $t \mapsto \mathbf{E}(X_t)$ and the covariance function

$$(s,t) \mapsto \mathbf{E}\left(\left(X_t - \mathbf{E}(X_t)\right)\left(X_s - \mathbf{E}(X_s)\right)\right).$$

Often in the definition of a Gaussian process it is assumed that the mean is zero.

An important property is the stationarity of a process. This means that the finite dimensional distributions do not change in time. We define in Definition 1.7, when stochastic processes have the same infinite dimensional distributions. We state the definition and the theorem of Dym and McKean [18]. In this definition the mean is assumed to be zero.

Definition 1.2. The Gaussian process $\{X_t : t \in \mathbb{R}\}$ of zero mean is called *stationary* if the process $\{X_{T+t} : t \in \mathbb{R}\}$ has the same finite dimensional distributions as the process $\{X_t : t \in \mathbb{R}\}$, for any $T \in \mathbb{R}$. In other words, in the stationarity case the probability

$$\mathbf{P}(\bigcap_{i=1}^{n}\{a_i \leq X_{t_i+T} \leq b_i\})$$

does not depend on $T \in \mathbb{R}$ for any $n \in \mathbb{N}$.

There is also a weaker form of stationarity for all processes (not only Gaussian). If the process is not necessarily stationary but its mean and variance are constants and the covariance depends only on the difference of the time, then we say that the process is *second order stationary.* This definition is used, for example, in Cowpertwait and Metcalfe [13]. In the next theorem we actually show that every stationary Gaussian process is also second order stationary. Thus, another way to state the stationarity for Gaussian processes is

Theorem 1.3. *The Gaussian process $\{X_t : t \in \mathbb{R}\}$ of zero mean is stationary if and only if the covariance $\mathbf{E}(X_{t_1+T}X_{t_2+T})$ does not depend on T, for any $t_1, t_2 \in \mathbb{R}$.*

Proof. Let $\{X_t : t \in \mathbb{R}\}$ be a Gaussian process of zero mean. If the process is stationary then obviously the covariance does not depend on T.

Conversely, if we assume that

$$\mathbf{E}(X_{t_1+T}X_{t_2+T}) = \mathbf{E}(X_{t_1}X_{t_2})$$

then

$$\mathbf{P}\left(\bigcap_{i=1}^{n}\{a_i \leq X_{t_i+T} \leq b_i\}\right) = \mathbf{P}\left(\bigcap_{i=1}^{n}\{a_i \leq X_{t_i} \leq b_i\}\right)$$

for any $n \in \mathbb{N}$, since in the Gaussian case, the covariance function determines distribution uniquely. $\square$

We denote the *covariance function* by

$$\mathrm{cov}(X_t, X_s) := \mathbf{E}\left((X_t - \mathbf{E}X_t)(X_s - \mathbf{E}X_s)\right).$$

And we define the *covariance matrix or covariance-variance matrix* of two random vectors $X := (X_{t_1}, X_{t_2}, \cdots, X_{t_n})$ and $Y := (Y_{s_1}, Y_{s_2}, \cdots, Y_{s_n})$ for $t_i, s_j \in \mathbb{R}, i, j = 1, \ldots, n$ by $[a_{ij}]_{n\times n}$, with the general element

$$a_{ij} = \mathbf{E}\left((X_{t_i} - \mathbf{E}X_{t_i})(Y_{t_j} - \mathbf{E}Y_{t_j})\right).$$

Theorem 1.4. *Let $\{X_t : t \in \mathbb{R}\}$ be a Gaussian process. Then for any $n \in \mathbb{N}$ and every $t_1, \ldots, t_n \in \mathbb{R}$ the covariance matrix of the multivariate Gaussian random vector $(X_{t_1}, \ldots, X_{t_n})$ is non-negative definite.*

Proof. Let $\{X_t : t \in \mathbb{R}\}$ be a Gaussian process and therefore the random vector $(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$ is Gaussian for any $t_1, t_2, \ldots, t_n \in \mathbb{R}$. We write its covariance matrix as

$$\mathbf{Q} := \begin{bmatrix} \mathbf{Q}_{11} & \cdots & \mathbf{Q}_{1n} \\ & \vdots & \\ \mathbf{Q}_{n1} & \cdots & \mathbf{Q}_{nn} \end{bmatrix},$$

where

$$\mathbf{Q}_{jk} := \mathbf{E}\left((X_{t_j} - \mathbf{E}(X_{t_j}))(X_{t_k} - \mathbf{E}(X_{t_k}))\right).$$

Since every covariance matrix $\mathbf{Q}$ is symmetric and symmetric matrices are orthogonally diagonalizable there exists a $n \times n$ matrix C such that the matrix $C^T\mathbf{Q}C$, denoted by B, is a diagonal matrix and the eigenvalues of $\mathbf{Q}$ are located in the main diagonal of B.

We consider the diagonal matrix B

$$\begin{aligned} B &= C^T\mathbf{Q}C \\ &= C^T\mathbf{E}\left([X_{t_1}, X_{t_2}, \ldots, X_{t_n}]^T[X_{t_1}, X_{t_2}, \ldots, X_{t_n}]\right)C \\ &= \mathbf{E}\left(C^T[X_{t_1}, X_{t_2}, \ldots, X_{t_n}]^T[X_{t_1}, X_{t_2}, \ldots, X_{t_n}]C\right) \\ &= \mathbf{E}\left(([X_{t_1}, X_{t_2}, \ldots, X_{t_n}]C)^T\,[X_{t_1}, X_{t_2}, \cdots, X_{t_n}]C\right), \end{aligned}$$

where $Y = [X_{t_1}, X_{t_2}, \cdots, X_{t_n}]C$, is Gaussian being a linear combination of Gaussian random variables X_{t_i}, $i = 1, \ldots, n$ and therefore it is a Gaussian random vector. Thus, B is the covariance matrix of the Gaussian vector Y. Since B is a diagonal matrix it actually consists of the variances of Y and we write that $B = \mathbf{V}ar(Y)$. Let $\mathbf{v}$ be a row vector $\mathbf{v} = \mathbf{u}C^T$. Applying previous statements, we may write as follows

$$\begin{aligned} \mathbf{v}\mathbf{Q}\mathbf{v}^T &= \mathbf{u}C^T\mathbf{Q}C\mathbf{u}^T \\ &= \mathbf{u}B\mathbf{u}^T \\ &= \mathbf{u}\mathbf{V}ar(Y)\mathbf{u}^T \\ &= \mathbf{V}ar(\mathbf{u}\cdot Y) \\ &\geq 0. \end{aligned}$$

Hence $\mathbf{v}\mathbf{Q}\mathbf{v}^T = \displaystyle\sum_{j,k=1}^{d} Q_{jk}v_iv_j \geq 0$, and therefore the covariance matrix is non-negative definite. □

We represent some important definitions of continuity and equality of stochastic processes.

Definition 1.5. A process $\{X_t : t \in \mathbb{R}\}$ is called L^2-*continuous at* t_0, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that the property

$$\mathbf{E}\left(|X_t - X_{t_0}|^2\right) < \varepsilon$$

holds for all $|t - t_0| < \delta$.

Dealing with stochastic processes, we often need their continuous versions (modifications). The following definition of a version may be found in Klebaner [32].

Definition 1.6. Two stochastic processes $\{X_t : t \in \mathbb{R}\}$ and $\{Y_t : t \in \mathbb{R}\}$ are called *versions* (modifications) of each other if

$$\mathbf{P}\left(X_t = Y_t\right) = 1, \text{ for all } t \geq 0.$$

Note that two stochastic processes may be versions of each other although one of them is continuous, but the other is not. However, if X is a version of Y, then X and Y have the same finite dimensional distributions. The definition is in Karatzas and Shreve [31].

Definition 1.7. $\mathbb{R}^d$-valued Stochastic processes $X = \{X_t : t \geq 0\}$ and $Y = \{Y_t : t \geq 0\}$ have the *same finite dimensional distributions* if, for any integer $n \geq 1$, real numbers $0 \leq t_1 < t_2 < \cdots < t_n < \infty$ and $A \in \mathcal{B}(\mathbb{R}^{nd})$, where $\mathcal{B}(\mathbb{R}^{nd})$ is the smallest σ-field containing all open sets of $\mathbb{R}^d$, we have:

$$\mathbf{P}\left((X_{t_1}, \ldots, X_{t_n}) \in A\right) = \mathbf{P}\left((Y_{t_1}, \ldots, Y_{t_n}) \in A\right).$$

If X and Y have the same finite dimensional distributions we use the notation

$$\{X_t : t \geq 0\} \stackrel{d}{=} \{Y_t : t \geq 0\}.$$

There is also a stricter requirement for the identity of the two processes:

Definition 1.8. Two processes $\{X_t : t \geq 0\}$ and $\{Y_t : t \geq 0\}$ are *indistinguishable*, if

$$\mathbf{P}\left(X_t = Y_t, \text{ for all } t \geq 0\right) = 1.$$

The indistinguishability of the processes means the sample paths of the processes are almost surely equal. The indistinguishability of processes requires slightly more than that the property of processes be versions of the each other, since indistinguishable processes are versions of each other, but the converse is not necessarily true. See, for example, Capasso and Bakstein [9].

If the processes X and Y are defined on the same state space but different probability space, we can define whether they have the same finite dimensional distribution, see, for example, [31].

Definition 1.9. Let $X = \{X_t : t \geq 0\}$ and $Y = \{Y_t : t \geq 0\}$ be stochastic processes defined on probability spaces $(\Omega, \mathcal{F}, \mathbf{P})$ and $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{P}})$, respectively, and having the same state space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. Stochastic processes X and Y have the *same finite dimensional distributions* if, for any integer $n \geq 1$, real numbers $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n < \infty$ and $A \in \mathcal{B}(\mathbb{R}^{nd})$, we have:

$$\mathbf{P}\left((X_{t_1}, \ldots, X_{t_n}) \in A\right) = \widetilde{\mathbf{P}}\left((Y_{t_1}, \ldots, Y_{t_n}) \in A\right).$$

We emphasize that if there is a continuous version, we use that. For this reason we rewrite the special continuity theorem modified for the 1-dimensional time parameter from Borodin and Salminen [7, Ch.1, Sec.1]

Theorem 1.10 (The Kolmogorov[6] continuity criterion)**.** *Let $X = \{X_t : t \in [0, T]\}$ be a stochastic process. If there exist positive constants $\alpha > 0$, $\beta > 0$ and $M > 0$ such that*

$$\mathbf{E}\left(|X_t - X_s|^{\alpha}\right) \leq M|t - s|^{1+\beta}$$

for every $0 \leq s, t \leq T$, then X has a continuous version.

We recall the following technical lemma and after that we consider more continuity properties of a covariance and the L^2 continuity of a stationary Gaussian process. If the process $\{X_t : t \in \mathbb{R}\}$ of zero mean is stationary we denote

$$\mathbf{Q}(t_1, t_2) = \mathbf{E}(X_0 X_{t_2 - t_1}) = \mathbf{Q}(0, t_2 - t_1) =: \mathbf{Q}(t_2 - t_1).$$

[6] Andrey N. Kolmogorov (1903-1987), Russian mathematician.

Lemma 1.11. *Let $\{X_t : t \in \mathbb{R}\}$ be a stationary Gaussian process. Then*

$$\mathbf{E}\left((X_{t_2} - X_{t_1})^2\right) = 2\left(\mathbf{Q}(0) - \mathbf{Q}(\delta)\right),$$

for any $\delta = t_2 - t_1$ and $t_1, t_2 \in \mathbb{R}$.

Proof. We calculate

$$\begin{aligned}
\mathbf{E}\left((X_{t_2} - X_{t_1})^2\right) &= \mathbf{E}\left(X_{t_2}^2 - 2X_{t_1}X_{t_2} + X_{t_1}^2\right) \\
&= \mathbf{E}(X_{t_2}^2) - 2\mathbf{E}(X_{t_1}X_{t_2}) + \mathbf{E}(X_{t_1}^2) \\
&= \mathbf{Q}(t_2 - t_2) - 2\mathbf{Q}(t_2 - t_1) + \mathbf{Q}(t_1 - t_1) \\
&= 2\left(\mathbf{Q}(0) - \mathbf{Q}(\delta)\right).
\end{aligned}$$

□

Applying Lemma 1.11 we obtain more properties for the covariance function.

Theorem 1.12. *Let $\{X_t : t \in \mathbb{R}\}$ be a Gaussian process. If $\{X_t : t \in \mathbb{R}\}$ is stationary, then its covariance function $\mathbf{Q}$ is an even function. Moreover, if $\mathbf{Q}$ is continuous at zero, then it is continuous everywhere.*

Proof. The covariance is even, since

$$\mathbf{Q}(t-s) = \mathbf{E}(X_sX_t) = \mathbf{E}(X_tX_s) = \mathbf{Q}(-(t-s))).$$

Since $\mathbf{Q}$ is continuous at zero, using the Cauchy−Schwarz inequality, we obtain

$$\begin{aligned}
&\lim_{h\to 0} |\mathbf{Q}(t+h) - \mathbf{Q}(t)| \\
&\quad = \lim_{h\to 0} |\mathbf{E}\left(X_0X_{t+h}\right) - \mathbf{E}\left(X_0X_t\right)| \\
&\quad = \lim_{h\to 0} |\mathbf{E}\left(X_0\left(X_{t+h} - X_t\right)\right)| \\
&\quad \le \lim_{h\to 0} \left(\mathbf{E}\left(X_0^2\right)\right)^{\frac{1}{2}} \left(\mathbf{E}\left(X_{t+h} - X_t\right)^2\right)^{\frac{1}{2}} \\
&\quad = \lim_{h\to 0} \left(\mathbf{Q}(0)\right)^{\frac{1}{2}} \left(2\left(\mathbf{Q}(0) - \mathbf{Q}(-h)\right)\right)^{\frac{1}{2}} \\
&\quad = 0.
\end{aligned}$$

□

We are able to state the following lemma concerning the L^2 continuity. We first recall that the L^2 continuity of the process $\{X_t : t \in \mathbb{R}\}$ is uniform if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\mathbf{E}\left((X_{t_2} - X_{t_1})^2\right) < \varepsilon$$

for all $t_1, t_2 \in \mathbb{R}$ with $|t_2 - t_1| < \delta$. In other words, uniform L^2 continuity means that the continuity does not depend on $t_1, t_2 \in \mathbb{R}$.

Lemma 1.13. *The stationary Gaussian process $X := \{X_t : t \in \mathbb{R}\}$ is uniformly L^2 continuous if it is continuous at zero.*

Proof. Assume that X is a stationary Gaussian process and let $\mathbf{Q}$ be its covariance function. We assume that $\mathbf{Q}$ is continuous at zero. By the previous theorem it is continuous everywhere. Let $t_1, t_2 \in \mathbb{R}$ and $\varepsilon > 0$. Since $\mathbf{Q}$ is continuous at zero there exists a $\delta > 0$ such that

$$|\mathbf{Q}(0) - \mathbf{Q}(u)| < \frac{\varepsilon}{2},$$

for $|u| < \delta$. Denoting $u = t_2 - t_1$ and applying Lemma 1.11, we infer

$$\mathbf{E}\left((X_{t_2} - X_{t_1})^2\right) < \varepsilon,$$

hence the process is uniformly L^2 continuous. □

Corollary 1.14. *The covariance function* $\mathbf{Q}$ *of a stationary Gaussian process* X *attains its maximum at zero.*

Proof. Applying the technical Lemma 1.11 and the fact that $\mathbf{E}\left((X_{t_2} - X_{t_1})^2\right)$ is always non-negative, we infer that $\mathbf{Q}(0) \geq \mathbf{Q}(\delta)$, for all δ and therefore the greatest value of $\mathbf{Q}$ is attained at zero. □

Definition 1.15. Let $X = \{X_t : t \geq 0\}$ be a stochastic process and $T \in \mathbb{R}_+$. If for all $T > 0$ there exists some $\beta > 0$ and a finite random variable $K_T(\omega)$ satisfying the condition

$$\sup_{s,t<T; s\neq t} \frac{|X_t(\omega) - X_s(\omega)|}{|t-s|^\beta} \leq K_T(\omega)$$

for almost all ω, then X is called *locally Hölder continuous of the order* β.

There is the following connection between the Kolmogorov criterion and the Hölder continuity. If there exist strictly positive α and β such that

$$\mathbf{E}|X_t - X_s|^\alpha \leq M|t-s|^{1+\beta}$$

then the process X has a Hölder continuous version of any order $\gamma < \frac{\beta}{\alpha}$. This remark can be found, for example, in Revuz and Yor [52, Theorem 2.1, p.26].

1.2 Self-similarity

Sometimes a process looks the same as the original one, although the scale, on which it is looked at, is changed from macroscopic to microscopic. This phenomenon is called self-similarity and it is known from nature. For example, the branching of trees is a self-similar process. Another visual example is the romanesco broccoli, which contributes to understanding the meaning of self-similarity.

The process in Figure 1.2 is Brownian motion $\{B_t : t \geq 0\}$ and it is still perhaps the most famous example of self-similarity.

In the middle of the 20th century Hurst[7] studied changes of the elevation of the water in the
time scale. Hurst built up a new statistical method, R/S analysis, which has connections with long-range dependent processes, see, for example, Hurst [23], [24] and [25]. When

[7]Harold E. Hurst (1880-1978), British hydrologist.

Figure 1.1: Romanesco broccolli, the picture is from Walker [57]

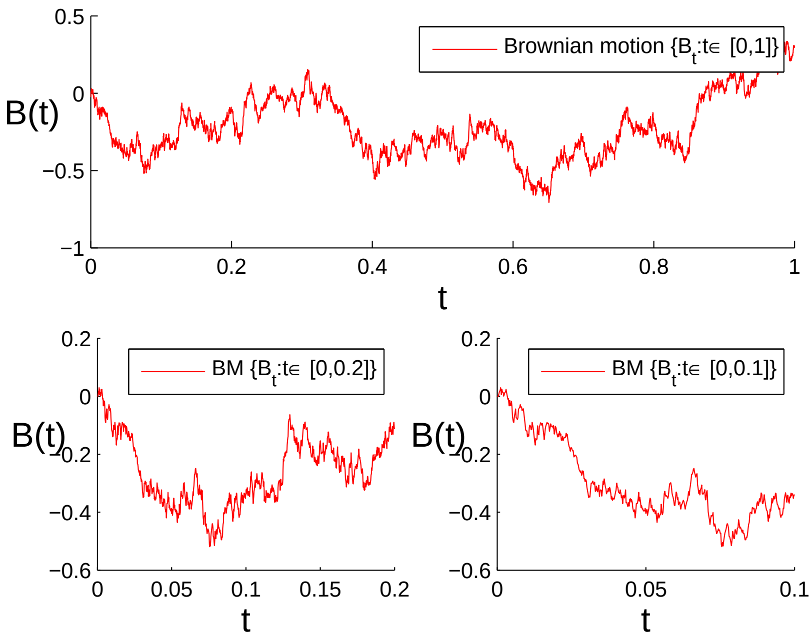

Figure 1.2: Sample paths of one Brownian motion, $t \in [0, 1]$, $t \in [0, 0.2]$ and $t \in [0, 0.1]$

Mandelbrot[8] and Van Ness [38] started to study fractional Brownian motion, they named the constant H as a *Hurst constant* in his honour.

[8]Benoît B. Mandelbrot (1924-2010), Polish-born, French and American mathematician

Lamperti[9] investigated mathematically the same kind of processes as Hurst. He studied the convergence of stochastic processes [34] and recognized stationarity in many of those processes. In [35] Lamperti introduced *semi-stable processes* that satisfy the same scaling property as the self-similar processes.

Definition 1.16. A stochastic process $\{X_t : t \in \mathbb{R}\}$ is called H-*self-similar*, if the stochastic processes $\{X_{\alpha t} : t \in \mathbb{R}\}$ and $\{\alpha^H X_t : t \in \mathbb{R}\}$ have the same finite dimensional distributions for all $\alpha \in \mathbb{R}$ and for $H \in (0, 1)$, that is,

$$\{X_{\alpha t} : t \in \mathbb{R}\} \stackrel{d}{=} \left\{\alpha^H X_t : t \in \mathbb{R}\right\}.$$

We study many processes that exhibit the property of self-similarity. The most common self-similar process is Brownian motion studied in Section 1.4.1.

1.3 Asymptotic behaviour and long-range dependence

We also need some properties of the asymptotic behaviour of the processes when studying long and short-range dependencies.

Thus, we present the familiar symbol "Ordo" to consider the growing rates of functions.

Remark 1.17. Let the functions f and g be defined in the same neighbourhood N_0 on $x \in \mathbb{R} \cup \{-\infty, \infty\}$. If there exists strictly positive k such that

$$\left|\frac{f(x)}{g(x)}\right| \leq k,$$

for any $x \in N_0$, then we define

$$f(x) = \mathbf{O}(g(x)) \text{ as } x \to x_0.$$

The idea of this notation is that f increases more slowly or decreases more rapidly than some multiple of g. If there exist strictly positive k_1 and k_2 such that

$$k_1 \leq \left|\frac{f(x)}{g(x)}\right| \leq k_2,$$

for any $x \in N_0$, then f has the same asymptotic behaviour as g, and they both increase or decrease at the same rate. In this case we use the notation

$$f(x) = \theta(g(x)) \text{ as } x \to x_0.$$

In the literature there are many definitions of long-range dependence. These all have the same idea or contents: If the process $\{X_t : t \geq 0\}$ is long-range dependent then its covariance vanishes slowly, in particularly not exponentially.

Definition 1.18. A stationary second order process $\{X_t : t \in \mathbb{R}\}$ or a sequence $\{X_n : n \in \mathbb{N}\}$ of zero mean is called *long-range dependent* if

$$\sum_{n=1}^{\infty} cov(X_1, X_n) = \sum_{n=1}^{\infty} \mathbf{E}(X_1 X_n)$$

diverges. If the sum converges, then the process or the sequence is called *short-range dependent*.

[9] John W. Lamperti, American mathematician.

Remark 1.19. We use Definition 1.18, since it is easy to understand and apply. However, a stationary second order stochastic process $X = \{X_n : n = 0, 1, 2, \dots\}$ of mean zero is, sometimes called

- *long-range dependent* if there exists $\alpha \in (0,1)$ and a constant $C > 0$ such that $\lim_{n\to\infty} \frac{\rho_X(n)}{C\, n^{-\alpha}} = 1$, where $\rho_X(n) := \mathbf{E}\left(X_i X_{i+n}\right)$, for any non-negative integer i,

and

- *short-range dependent* if $\lim_{k\to\infty} \sum_{n=0}^{k} \rho_X(n)$ exists.

This kind of definition, for example, stated in Beran [4, p. 6 and p. 42] and is occasionally more convenient to use than our Definition 1.18.

Definition 1.18 is not always equivalent to the above definition from [4]. Indeed, if we consider the case where

$$\mathbf{E}(X_{n+k}X_k) = \frac{1}{n},\ n = 1, 2, \dots.$$

Then we have harmonic series and

$$\sum_{n=1}^{\infty} \rho_X(n) = \lim_{k\to\infty} \sum_{n=1}^{k} \rho_X(n) = \lim_{k\to\infty} \sum_{n=1}^{k} \frac{1}{n} = \infty.$$

Hence, X is a long-range dependent according to Definition 1.18. But if $\alpha \in (0,1)$

$$\lim_{n\to\infty} \frac{\rho_X(n)}{Cn^{-\alpha}} = 0,$$

for any $C > 0$. This means that process X is not a long-range dependent according to the definition of the remark 1.19.

In this dissertation we will use Definition 1.18 for long-range dependence and short-range dependence.

1.4 Some important Gaussian processes

1.4.1 Brownian motion and OU processes

Brownian motion was invented by botanist Robert Brown in 1827 [8]. Calling Brown an inventor of Brownian motion may be venturesome, since there were some other researchers at his time studying the same field. Brown might have been the first to have published something about this phenomenon. Anyhow, he observed the pollen grains moving on the surface of water. The movement was random, and he did not understand the reason for this movement. First he thought that the reason was connected only to the organic particles, but later he also observed the same kind of movements with synthetic particles.

It took quite some time to find an explanation for this movement. Albert Einstein[10] recognized in 1905 that the reason for the movement is thermodynamic. But actually

[10] Albert Einstein (1879-1955), German theoretical physicist

Luis Bachelier[11] was the first person to model Brownian motion. He used Brownian motion to model stock prices on the Paris Stock Exchange in 1900. In mathematics Brownian motion is also often called the Wiener process, since Norbert Wiener[12] proved the existence of Brownian motion defined as:

Definition 1.20. A real valued stochastic process $\{B_t : t \geq 0\}$ is called the *standard Brownian motion* (Bm) starting at zero, if the following properties holds

(i) $B_0 = 0$ a.s.;

(ii) $\mathbf{E}\left(B_{t_i} - B_{t_{i-1}}\right)^2 = t_i - t_{i-1}$;

(iii) for any $0 = t_0 < t_1 < \cdots < t_n$ the increments

$$B_{t_n} - B_{t_{n-1}}, B_{t_{n-1}} - B_{t_{n-2}}, \ldots, B_{t_1} - B_{t_0}$$

are independent and normally distributed with

$$\mathbf{E}\left(B_{t_i} - B_{t_{i-1}}\right) = 0.$$

The item (ii) from Definition 1.20 implies, that B has continuous paths a.s. We recall the definition of the Markov process in [32] and Ornstein−Uhlenbeck process [50], since they both involved Brownian motion.

Definition 1.21. If for any t and $s > 0$, the conditional distribution of X_{t+s} given σ-field $\mathcal{F}_t$ is the same as the conditional distribution of X_{t+s} given X_t, that is, for all $y \in \mathbb{R}$

$$\mathbf{P}(X_{t+s} \leq y|\mathcal{F}_t) = \mathbf{P}(X_{t+s} \leq y|X_t) \text{ a.s. },$$

then X is a *Markov process.*

There are two different ways to construct the OU process; either via a time and space transformation, which is also called *the Doob transformation* or as a solution of a stochastic differential equation of which the driving process is the standard Brownian motion. First we present definition of the Doob transformation.

Definition 1.22. Let $\{B_t : t \geq 0\}$ be the standard Brownian motion and $\alpha > 0$. Then the process $\{V_t : t \in \mathbb{R}\}$

$$V_t := \mathrm{e}^{-\alpha t} B_{\frac{\mathrm{e}^{2\alpha t}}{2\alpha}},$$

is called the *Ornstein − Uhlenbeck process (OU).*

The preceding well-known construction of the OU process is due to Doob [16] and it is a deterministic time and space transformation of the standard Brownian motion. The covariance of V is

$$\begin{aligned}
\mathbf{E}(V_t V_s) &= \mathrm{e}^{-\alpha t - \alpha s} \mathbf{E}\left(B_{\frac{\mathrm{e}^{2\alpha t}}{2\alpha}} B_{\frac{\mathrm{e}^{2\alpha s}}{2\alpha}}\right) \\
&= \mathrm{e}^{-\alpha t - \alpha s} \min\left(\frac{\mathrm{e}^{2\alpha t}}{2\alpha}, \frac{\mathrm{e}^{2\alpha s}}{2\alpha}\right) \\
&= \frac{1}{2\alpha}\mathrm{e}^{-\alpha(t-s)}, \quad \text{if } t \geq s.
\end{aligned}$$

[11] Louis J-B. A. Bachelier (1870-1946), French mathematician
[12] Norbert Wiener (1894-1964), American mathematician

Since the covariance depends only on the time difference, therefore the process is stationary.

Secondly we construct an OU process, as a strong and unique solution to the Langevin stochastic differential equation. The solution of the linear first order differential equation is unique if we have initial value $X_a = b$.

Definition 1.23. Let $\{B_t : t \geq 0\}$ be the standard Brownian motion and $\alpha > 0$. The solution of the stochastic differential equation

$$dU_t = -\alpha U_t dt + dB_t, \tag{1.2}$$

is also called the *Ornstein–Uhlenbeck process (OU1).* The solution is

$$U_t = \mathrm{e}^{-\alpha t} \left(x + \int_0^t \mathrm{e}^{\alpha s} dB_s \right), \ t \geq 0, \tag{1.3}$$

where x is the random initial value of U.

Recall the properties of a strong solution from Øksendal [49, p.66]. We first define some terms: $\mathcal{F}_\infty$ is the smallest σ-algebra contains $\bigcup_{t>0} \mathcal{F}_t$ and $\{B_t : t \geq 0\}$ is 1-dimensional Brownian motion.

Theorem 1.24. *Let $T > 0$ and $b : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ be measurable functions satisfying*

$$|b(t, x)| + |\sigma(t, x)| \leq C(1 + |x|); \quad x \in \mathbb{R}^n, t \in [0, T]$$

for some constant C, (where $|\sigma|^2 = \sum |\sigma_{ij}|^2$) and such that

$$|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq D|x - y|; \quad x, y \in \mathbb{R}^n, t \in [0, T]$$

for some constant D. Let Z be a random variable which is independent of the σ-algebra $\mathcal{F}_\infty$ generated by $\{B_s : s \geq 0\}$ and such that

$$\mathbf{E}(|Z|^2) < \infty.$$

Then the stochastic differential equation

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t,$$

where $0 \leq t \leq T, X_0 = Z$, has a unique t-continuous solution $X_t(\omega)$ with the property that $X_t(\omega)$ is adapted to the filtration $\mathcal{F}_t^Z$ generated by Z and $B_s; s \leq t$ and

$$\mathbf{E} \left(\int_0^T |X_t|^2 dt \right) < \infty.$$

This kind of solution $X = \{X_t : t \in [0, T]\}$ from Theorem 1.24 is called a *strong* solution, see, for example, [49, p. 70]. We know that every linear stochastic differential equation with constant coefficients has a unique strong solution at every interval $[0, T]$, see, for example, Mikosch [42, p. 138].

In fact the solution of stochastic differential equation (1.3), The Ornstein–Uhlenbeck process, in Definition 1.23 is strong and unique, but it is not yet stationary. First we extend it to the whole time space and then define the initial value to make it stationary.

Definition 1.25. Let $B^{(-)} = \{B_t^{(-)} : t \geq 0\}$ be another Brownian motion, independent of B, also starting from 0. When $t \in \mathbb{R}$, we set *the two-sided Brownian motion*

$$\widehat{B}_t := \begin{cases} B_t\,, & t \geq 0, \\ B_{-t}^{(-)}\,, & t < 0. \end{cases}$$

If in Definition 1.23 the variable x is equal to ξ

$$\xi := \int_{-\infty}^{0} \mathrm{e}^{\alpha s} d\widehat{B}_s,$$

then ξ is a normally distributed random variable with the mean 0 and the variance $\frac{1}{2\alpha}$.

Theorem 1.26. *The Ornstein-Uhlenbeck process U, defined by*

$$U_t = \mathrm{e}^{-\alpha t} \int_{-\infty}^{t} \mathrm{e}^{\alpha s} d\widehat{B}_s, \tag{1.4}$$

is the stationary solution of (1.2).

Proof. From the considerations above it is clear that $\{U_t : t \geq 0\}$ solves (1.3) with

$$U_0 = x = \int_{-\infty}^{0} \mathrm{e}^{\alpha s} d\widehat{B}_s.$$

To prove stationarity, we compute as follows. The covariance of the process U may be computed as

$$\begin{aligned} \mathbf{Q}(t-s) &= \mathbf{E}(U_t U_s) \\ &= \mathbf{E}\left(\left(\mathrm{e}^{-\alpha t} \int_{-\infty}^{t} \mathrm{e}^{\alpha r} d\widehat{B}_r\right)\left(\mathrm{e}^{-\alpha s} \int_{-\infty}^{s} \mathrm{e}^{\alpha r} d\widehat{B}_r\right)\right) \\ &= e^{-\alpha t}\mathrm{e}^{-\alpha s}\left(\mathbf{E}\left[\left(\int_{-\infty}^{s} \mathrm{e}^{\alpha r} d\widehat{B}_r\right)^2\right]\right. \\ &\qquad \left. + \mathbf{E}\left[\int_{s}^{t} \mathrm{e}^{\alpha r} d\widehat{B}_r \int_{-\infty}^{s} \mathrm{e}^{\alpha r} d\widehat{B}_r\right]\right), \end{aligned}$$

when $t > s$. If we use the Itô isometry in the first part of the sum and the independence of the increments of Brownian motion in the second one, we obtain

$$\begin{aligned} \mathbf{Q}(t-s) &= e^{-\alpha t}\mathrm{e}^{-\alpha s} \int_{-\infty}^{s} (\mathrm{e}^{\alpha r})^2 dr \\ &= \frac{e^{-\alpha(t-s)}}{2\alpha}. \end{aligned} \tag{1.5}$$

As we notice, the covariance is dependent only on the difference of time, so the process is stationary. □

We may also note that the covariance of U is the same as the covariance of V. Since these processes are both Gaussian processes, they are equally determined by their covariances. Thus the processes have the same finite dimensional distributions.

This connection was the first and the main one to lead me to study fractional OU processes. One of the main questions was whether or not they have the same finite dimensional distributions.

In this dissertation we study fractional Ornstein−Uhlenbeck processes and their properties. These processes are constructed similarly to OU processes but Brownian motion is replaced by fractional Brownian motion.

1.4.2 Fractional Brownian motion

In various problems, existing models are unsatisfactory. Brownian motion and the short-range dependent processes derived from it are not the best explanation for problems in data traffic or in the economical time series, for example.

A concept called fractional Brownian motion is an answer to many questions, it is neither a specified Brownian motion nor its contraction. Fractional Brownian motion is a generalization of Brownian motion in the sense that the when $H = \frac{1}{2}$ it coincides with Brownian motion. It belongs to the class of processes with a long memory, when $H > \frac{1}{2}$. Mandelbrot and Van Ness were the pioneers of studies of fractional Brownian motion [38]. Their definition for fBm is not so easy to use as the definition which we propose here.

Fractional Brownian motion is at the heart of the studies in this dissertation. We state two definitions of fractional Brownian motion.

The first definition by Mandelbrot and Van Ness can be found in [38]. Their definition of fractional Brownian motion uses an integral with respect to Brownian motion

$$B_H(0,\omega) = b_0$$

$$\begin{aligned} B_H(t,\omega) - B_H(0,\omega) \\ = \frac{1}{\Gamma(H+\frac{1}{2})} \Bigg(\int_{-\infty}^{0} & \left((t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}} \right) dB(s,\omega) \\ & + \int_0^t (t-s)^{H-\frac{1}{2}} dB(s,\omega) \Bigg), \end{aligned}$$

where Γ is the Gamma function.

The definition above is not so easy to use, and we state the more common definition, see, for example, Memin, Mishura and Valkeila [41], as follows.

Definition 1.27. Let $0 < H < 1$. *Fractional Brownian motion (fBm)* $\{Z_t : t \geq 0\}$ with Hurst parameter H is a centered Gaussian process with $Z_0 = 0$ and

$$\mathbf{E}(Z_t Z_s) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}),\ t,s \geq 0. \tag{1.6}$$

Theorem 1.28. *The fractional Brownian motion (fBm) $Z = \{Z_t : t \geq 0\}$ with a Hurst parameter $H \in (0,1)$ satisfies the properties*

(i) *Z is H-self-similar;*

(ii) *Z has stationary increments;*

(iii) $\mathbf{E}(Z_t) = 0$ *for any* $t \geq 0$*;*

(iv) $\mathbf{E}\left(Z_t^2\right) = t^{2H}$ *for any* $t \geq 0$.

Proof.

(i) Fractional Brownian motion is H-self-similar, i.e.,

$$\{Z_{\alpha t} : t \geq 0\} \overset{d}{=} \left\{\alpha^H Z_t : t \geq 0\right\} \text{ for any } \alpha > 0. \tag{1.7}$$

Indeed, the self-similarity of fBm follows from (1.6), since

$$\begin{aligned}
\mathbf{E}(Z_{\alpha t} Z_{\alpha s}) &= \frac{1}{2}((\alpha t)^{2H} + (\alpha s)^{2H} - |\alpha t - \alpha s|^{2H}) \\
&= \alpha^{2H} \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}) \\
&= \mathbf{E}(\alpha^H Z_t \alpha^H Z_s)
\end{aligned}$$

and the covariance determines the Gaussian distribution uniquely.

(ii) When $s_1 < s_2 < t_1 < t_2$, the relation (1.6) implies that

$$\begin{aligned}
&\mathbf{E}\left((Z_{t_2} - Z_{t_1})(Z_{s_2} - Z_{s_1})\right) \\
&= \frac{1}{2}((t_2 - s_1)^{2H} - (t_2 - s_2)^{2H} + (t_1 - s_2)^{2H} - (t_1 - s_1)^{2H}).
\end{aligned} \tag{1.8}$$

By (1.8) and Theorem 1.3 we observe that the increments of fractional Brownian motion are stationary, since

$$\begin{aligned}
&\mathbf{E}\left((Z_{t_2+h} - Z_{t_1+h})(Z_{s_2+h} - Z_{s_1+h})\right) \\
&= \frac{1}{2}((t_2 - s_1)^{2H} - (t_2 - s_2)^{2H} + (t_1 - s_2)^{2H} - (t_1 - s_1)^{2H}).
\end{aligned}$$

(iii) $\mathbf{E}(Z_t) = 0$, for any $t \geq 0$, since $\{Z_t : t \geq 0\}$ is a centered process.

(iv) For any $t \geq 0$, using (1.6)

$$\begin{aligned}
\mathbf{E}\left(Z_t^2\right) &= \mathbf{E}(Z_t Z_t) \\
&= \frac{1}{2}\left(t^{2H} + t^{2H}\right) \\
&= t^{2H}.
\end{aligned}$$

$\square$

In the literature there are also more axiomatic definitions for fractional Brownian motion. These definitions usually contain a collection of same kind of properties as we have in Theorem 1.28. A more axiomatic definition can be found, for example, in Norros, Valkeila and Virtamo [45]. The literature also has definitions of fractional Brownian motion where the time parameter t belongs to $\mathbb{R}$. See, for example, [40]. We chose our definition, because using it makes it simpler to construct stationary processes, as stochastic integrals, where fBm is a driving process or to make some time transformations with respect to fBm.

Next, we consider more properties of fractional Brownian motion.

Remark 1.29. From Theorem 1.28 (iv) we directly obtain that

$$\mathbf{E}(Z_0^2) = 0 \text{ and } \mathbf{E}(Z_1^2) = 1.$$

Using the first identity of the proof of Lemma 1.11, the property (iv) and the Kolmogorov continuity criterion (Theorem 1.10), we may notice that fractional Brownian motion Z has a continuous version, when $H > \frac{1}{2}$, since the increments of fractional Brownian motion are stationary and $Z_0 = 0$ and therefore

$$\mathbf{E}\left((Z_t - Z_s)^2\right) = \mathbf{E}\left(Z_{t-s}^2\right). \tag{1.9}$$

Indeed, we obtain the stronger result using the next lemma stated, for example, in Nualart [47].

Lemma 1.30. *Let $Z = \{Z_t : t \geq 0\}$ be fractional Brownian motion with Hurst parameter $H \in (0,1)$. Then*

$$\mathbf{E}\left((Z_t - Z_s)^{2k}\right) = \frac{(2k)!}{k!2^k}|t-s|^{2Hk},$$

for any integer $k \geq 1$.

Proof. Using (), we obtain

$$\begin{aligned}
\mathbf{E}\left((Z_t - Z_s)^{2k}\right) &= \mathbf{E}\left(Z_{t-s}^{2k}\right) \\
&= \mathbf{E}\left((|t-s|^H Z_1)^{2k}\right) \\
&= |t-s|^{2kH}\mathbf{E}\left(Z_1^{2k}\right),
\end{aligned}$$

where the second equation follows from the fact that fractional Brownian Motion is H self-similar. From the definition of fractional Brownian motion we conclude that $Z_1 \sim \mathcal{N}(0,1)$ and so its moment generating function is

$$\begin{aligned}
m_{Z_1}(p) &= \int_{-\infty}^{\infty} \mathrm{e}^{xp}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{2}}\,dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\left(\frac{x}{\sqrt{2}}-\frac{p}{\sqrt{2}}\right)^2}\mathrm{e}^{\frac{p^2}{2}}\,dx \\
&= \mathrm{e}^{\frac{p^2}{2}}.
\end{aligned}$$

Since $m_X^{(n)}(0) = \mathbf{E}(X^n)$, then

$$\begin{aligned}\mathbf{E}(Z_1^{2k}) = \left.\frac{d^{2k} m_{Z_1}(p)}{dp^{2k}}\right|_{p=0} &= (2k-1)(2k-3)(2k-5)\cdots 1 = (2k-1)!! \\ &= \frac{(2k-1)(2k-3)(2k-5)\cdots 1 \cdot 2k(2k-2)(2k-4)\cdots 2}{2k(2k-2)(2k-4)\cdots 2} \\ &= \frac{(2k)!}{2^k k!}.\end{aligned}$$

Hence

$$\begin{aligned}\mathbf{E}\left((Z_t - Z_s)^{2k}\right) &= |t-s|^{2kH}\mathbf{E}\left((Z_1)^{2k}\right) \\ &= \frac{(2k)!}{2^k k!}|t-s|^{2kH}.\end{aligned}$$

□

Theorem 1.31. *Fractional Brownian motion $\{Z_t : t \geq 0\}$ has a version with continuous paths.*

Proof. Applying Lemma 1.30 we notice that

$$\mathbf{E}\left((Z_t - Z_s)^{2k}\right) = \frac{(2k)!}{k!2^k}|t-s|^{2Hk},$$

for any integer $k \geq 1$. Choosing $k > \frac{1}{2H}$, we have a continuous version by the Kolmogorov continuity criterion (Theorem 1.10), for all $H \in (0,1)$. □

From now on Z is assumed to be continuous. Moreover, Z is locally Hölder continuous of the order β for any $\beta < H$, (see the definition of the Hölder continuity in Definition 1.15). The proof of the locally Hölder continuity of fractional Brownian motion, can be found, for example, in Sottinen [56]. We give a brief proof.

By Lemma 1.30 we infer

$$\mathbf{E}\left((Z_t - Z_s)^{2k}\right) = C|t-s|^{2kH} \tag{1.10}$$

and so in the Kolmogorov continuity criterion $\alpha = 2k$ and $\beta = 2kH - 1$.

Theorem 1.32. *Fractional Brownian motion has locally Hölder continuous version of any order $\beta < H$.*

Proof. Equation (1.10) states a connection between the Kolmogorov continuity criterion and the Hölder continuity (on page 8), as follows, if

$$\mathbf{E}|X_t - X_s|^\alpha \leq M|t-s|^{1+\beta}$$

holds, then the process X is Hölder continuous of any order $\gamma < \frac{\beta}{\alpha}$. Applying (1.10) the process Z is Hölder continuous of the order $H - \frac{1}{2k}$. When $k \to \infty$ we obtain that Z is Hölder continuous of the order $\gamma < H$. □

Proposition 1.33. *If a stochastic process is locally Hölder continuous of the order β and $\gamma < \beta$, then the stochastic process is locally Hölder continuous of the order γ.*

Proof. We assume that a stochastic process X is locally Hölder continuous of the order β and $I = [a, b] \subset \mathbb{R}_+$, then

$$\begin{aligned}\sup_{s,t\in I; s\neq t} \frac{|X_t - X_s|}{|t-s|^\gamma} &= \sup_{s,t\in I; s\neq t} \frac{|X_t - X_s|}{|t-s|^\beta}|t-s|^{\beta-\gamma} \\ &\leq \sup_{s,t\in I; s\neq t} \frac{|X_t - X_s|}{|t-s|^\beta}(b-a)^{\beta-\gamma} < M,\end{aligned}$$

since $|t - s| < b - a < \infty$ and $\beta - \gamma > 0$. □

Applying Mishura [43, Section 1.2.] we have the kernel representation of increments' covariance of fractional Brownian motion for $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, that is

$$\mathbf{E}\left((Z_{t_2} - Z_{t_1})(Z_{s_2} - Z_{s_1})\right) = \int_{s_1}^{s_2}\int_{t_1}^{t_2} (2H-1)H(u-v)^{2H-2}dudv, \tag{1.11}$$

if $s_1 < s_2 < t_1 < t_2$.

We state the following proposition

Proposition 1.34. *Let $\{Z_t : t \geq 0\}$ be fractional Brownian motion and $H \in (0, 1)$. Then*

- *if $H = \frac{1}{2}$ the increments are independent (Bm case),*
- *if $H > \frac{1}{2}$ the increments are positively correlated and*
- *if $H < \frac{1}{2}$ the increments are negatively correlated.*

Proof. We may prove this proposition using the covariance representation (1.8) of the increments of fractional Brownian motion and the kernel representation (1.11).

We recall that if the autocorrelation function $\rho_X(n) := \mathbf{E}(X_0 X_n)$ is positive, the process $\{X_t : t \in \mathbb{R}\}$ is positively correlated, and if it is negative, the process is negatively correlated.

Thus we consider the covariance of the increments, in the case $H = \frac{1}{2}$

$$\begin{aligned}&\mathbf{E}\left((Z_{t_2} - Z_{t_1})(Z_{s_2} - Z_{s_1})\right) \\ &= \frac{1}{2}((t_2 - s_1)^{2H} - (t_2 - s_2)^{2H} + (t_1 - s_2)^{2H} - (t_1 - s_1)^{2H}) \\ &= 0.\end{aligned}$$

Hence in the case of Brownian motion the increments are independent.

In the case $H \neq \frac{1}{2}$ we calculate the covariance of the increments of fractional Brownian motion to use the kernel representation as follows

$$\begin{aligned}\mathbf{E}\left((Z_{t_2} - Z_{t_1})(Z_{s_2} - Z_{s_1})\right) &= \int_{s_1}^{s_2}\int_{t_1}^{t_2} (2H-1)H(u-v)^{2H-2}dudv \\ &= (2H-1)H\int_{s_1}^{s_2}\int_{t_1}^{t_2}(u-v)^{2H-2}dudv.\end{aligned}$$

The integral is non-negative, since $s_1 < s_2 < t_1 < t_2$, and therefore

$$(2H-1)H\int_{s_1}^{s_2}\int_{t_1}^{t_2}(u-v)^{2H-2}dudv > 0, \text{ when } H > \frac{1}{2}$$

and

$$(2H-1)H\int_{s_1}^{s_2}\int_{t_1}^{t_2}(u-v)^{2H-2}dudv < 0, \text{ when } H < \frac{1}{2}$$

completing the proof. □

Recall that Theorem 1.28 (ii) states stationarity of the increments of fractional Brownian motion $Z = \{Z_t : t \geq 0\}$, however, Z itself is not stationary. Hence the natural step is to define the increment process of fractional Brownian motion.

Definition 1.35. A stationary second order stochastic process $\{I_Z(n) : n \in \mathbb{N}_0\}$ is called *the increment process of fractional Brownian motion or the fractional Gaussian noise*, if

$$I_Z = \{Z_{n+1} - Z_n : n = 0, 1, 2 \ldots\}, \tag{1.12}$$

and $\{Z_t : t \geq 0\}$ is fractional Brownian motion.

The kernel representation (1.11) implies the following result. It can also be found, for example, in [56, p. 9], but we present a brief proof.

Proposition 1.36. *Let I_Z be the increment process of fractional Brownian motion. Then the autocorrelation function $\rho_{I_Z}(n)$ satisfies*

$$\begin{aligned}\rho_{I_Z}(n) &:= \mathbf{E}\left(I_Z(0)I_Z(n)\right) \\ &= \mathbf{E}\left(Z_1(Z_{n+1} - Z_n)\right)\end{aligned}$$

and holds

$$\rho_{I_Z}(n) = \theta(n^{2H-2}), \tag{1.13}$$

when n tends to ∞.

Proof. Note first that

$$\begin{aligned}\rho_{I_Z}(n) &= \mathbf{E}\left(I_Z(0)I_Z(n)\right) \\ &= \mathbf{E}\left((Z_1 - Z_0)(Z_{n+1} - Z_n)\right) \\ &= \mathbf{E}\left(Z_1(Z_{n+1} - Z_n)\right).\end{aligned}$$

Then we use the kernel representation (1.11) to obtain

$$\begin{aligned}\mathbf{E}\left(Z_1(Z_{n+1}-Z_n)\right) &= \int_0^1\int_n^{n+1}(2H-1)H(u-v)^{2H-2}dudv \\ &= \int_0^1\int_0^1(2H-1)H(t+n-v)^{2H-2}dtdv.\end{aligned}$$

Applying twice the Mean Value Theorem we see that there exists $d \in (0,1)$ and $c \in (0,1)$, such that

$$\begin{aligned}\rho_{I_Z}(n) &= \int_0^1 (2H-1)H(d+n-v)^{2H-2}dv \\ &= (2H-1)H(d+n-c)^{2H-2}.\end{aligned}$$

Denoting $a := d-c$, we have

$$\rho_{I_Z}(n) = (2H-1)H(n+a)^{2H-2},$$

where $-1 < a < 1$. After some approximations, we obtain

$$(n+a)^{2H-2} \leq (n-1)^{2H-2} \leq 2n^{2H-2}$$

and

$$(n+a)^{2H-2} \geq (n+1)^{2H-2} \geq \frac{1}{2}n^{2H-2}$$

and therefore

$$\rho_{I_Z}(n) = (2H-1)H(n+a)^{2H-2} = \theta(n^{2H-2}),$$

when n is large and tending to infinity. □

Since we have an asymptotic approximation of the autocorrelation function, we may consider if the preceding increment process is long-range dependent or not. To study that we may use Definition 1.18 and (1.13).

Proposition 1.37. *The increment process I_Z of fractional Brownian motion Z is*

- *long-range dependent if $H > \frac{1}{2}$,*
- *short-range dependent if $H < \frac{1}{2}$.*

Proof. From the proof of Proposition 1.36 we obtain

$$|\rho_{I_Z}(n)| \leq H(1-2H)(n-1)^{2H-2},$$

for $H < \frac{1}{2}$. We consider the sum of the absolute values of covariance and obtain

$$\sum_{n=1}^{\infty} |H(2H-1)(n-1)^{2H-2}| = \sum_{n=0}^{\infty} |H(2H-1)n^{2H-2}| < \infty.$$

Using direct comparison we infer that

$$\sum_{n=1}^{\infty} \rho_{I_Z}(n) < \infty,$$

for $H < \frac{1}{2}$. Applying again the proof of Proposition 1.36 we obtain

$$\rho_{I_Z}(n) \geq (2H-1)H(n+1)^{2H-2},$$

for $H > \frac{1}{2}$. We consider the sum and obtain

$$\sum_{n=1}^{\infty} H(2H-1)(n+1)^{2H-2} = \sum_{n=2}^{\infty} H(2H-1)n^{2H-2} = \infty,$$

when $2H-2 > -1$. Thus

$$\sum_{n=1}^{\infty} H(2H-1)(n+1)^{2H-2}$$

diverges for $H > \frac{1}{2}$ and therefore the sum $\sum\limits_{n=1}^{\infty} \rho_{I_Z}(n)$ also diverges by direct comparison. We conclude, in the case $H < \frac{1}{2}$ that the sum $\sum\limits_{n=1}^{\infty} \rho_{I_Z}(n)$ is finite and, in the case $H > \frac{1}{2}$ that the sum $\sum\limits_{n=1}^{\infty} \rho_{I_Z}(n)$ is infinite. Using Definition 1.18 we conclude that the increment process I_Z is long-range dependent if $H > \frac{1}{2}$, and short-range dependent if $H < \frac{1}{2}$. $\square$

1.4.3 The Fractional Ornstein–Uhlenbeck process, fOU(1)

To define a fractional Ornstein−Uhlenbeck process as a solution of the Langevin SDE, we proceed similarly as in the usual OU case, but we use fractional Brownian motion instead of Brownian motion. Consider the following linear stochastic differential equation

$$dU_t^{(Z,\alpha)} = -\alpha U_t^{(Z,\alpha)} dt + dZ_t, \tag{1.14}$$

where $\alpha > 0$. Using integrating factor $\mathrm{e}^{\alpha t}$ we obtain the solution as follows

$$U_t^{(Z,\alpha)} = \mathrm{e}^{-\alpha t}\left(x + \int_0^t \mathrm{e}^{\alpha s} dZ_s\right), \tag{1.15}$$

where x is some random initial value. This pathwise Riemann−Stieltjes integral does indeed exist (see, for example, Cheridito, Kawaguchi and Maejima [12]). For positive values s the following integration by parts holds

$$\int_0^s \mathrm{e}^{\alpha u} dZ_u = \mathrm{e}^{\alpha s} Z_s - \int_0^s \alpha \mathrm{e}^{\alpha u} Z_u du. \tag{1.16}$$

We define the two-sided fractional Brownian motion next:

Definition 1.38. Let $Z^{(-)} = \{Z_t^{(-)} : t \geq 0\}$ be another fractional Brownian motion, independent of Z, also starting from 0. When $t \in \mathbb{R}$, *the two-sided fractional Brownian motion* is

$$\widehat{Z}_t := \begin{cases} Z_t\,, & t \geq 0, \\ Z_{-t}^{(-)}\,, & t < 0. \end{cases}$$

Lemma 1.39. *Let $\widehat{Z} = \{\widehat{Z}_t : t \in \mathbb{R}\}$ be the two-sided fractional Brownian motion. Then the formula*

$$\xi := \int_{-\infty}^{0} \mathrm{e}^{\alpha s} d\widehat{Z}_s \tag{1.17}$$

yields a well-defined random variable.

Proof. We first extend (1.16) for negative values by

$$\int_s^0 \mathrm{e}^{\alpha u} d\widehat{Z}_u = -\mathrm{e}^{\alpha s}\widehat{Z}_s - \int_s^0 \alpha \mathrm{e}^{\alpha u}\widehat{Z}_u du. \tag{1.18}$$

Then we have to prove that the limit on the right-hand side of (1.18) exists, when $s \to -\infty$. If we prove that the integral on the right-hand side in (1.17) exists and $\mathrm{e}^{\alpha s}\widehat{Z}_s$ tends to zero, then the random variable ξ is well defined.

If we define

$$\left\{ Z_t^{(o)} := t^{2H}\widehat{Z}_{-\frac{1}{t}} : t > 0 \right\},$$

then $Z_t^{(o)}$ is a centered Gaussian process and has the same covariance kernel as fractional Brownian motion, since the covariance of $\left\{ t^{2H}\widehat{Z}_{-\frac{1}{t}} : t > 0 \right\}$ is given by

$$\begin{aligned} \mathbf{E}\left(t^{2H}\widehat{Z}_{-\frac{1}{t}} s^{2H}\widehat{Z}_{-\frac{1}{s}} \right) &= \frac{1}{2} t^{2H}s^{2H}\left(\left| -\frac{1}{t} \right|^{2H} + \left| -\frac{1}{s} \right|^{2H} - \left| -\frac{1}{t} + \frac{1}{s} \right|^{2H} \right) \\ &= \frac{1}{2}\left(\left| -\frac{ts}{t} \right|^{2H} + \left| -\frac{ts}{s} \right|^{2H} - \left| -\frac{ts}{t} + \frac{ts}{s} \right|^{2H} \right) \\ &= \frac{1}{2}\left(|s|^{2H} + |t|^{2H} - |-s+t|^{2H} \right), \end{aligned}$$

which is actually the same as the covariance of fractional Brownian motion $\{Z_t : t > 0\}$. We obtain

$$\left\{ Z_t^{(o)} : t > 0 \right\} \stackrel{d}{=} \left\{ \widehat{Z}_{-t} : t > 0 \right\}.$$

We may prove that both of these processes

$$\{\widehat{Z}_{-t} : t > 0\} \text{ and } \{Z_t^{(o)} : t > 0\}$$

goes to the zero when t tends to the zero. By Definition 1.38 we know that $\widehat{Z}_0 = 0$. Hence using the Borel-Cantelli Lemma we first prove $\limsup_{t \to 0+} Z_t^{(o)} = 0$. Let $\varepsilon > 0$. Since fractional Brownian motion has continuous paths, we have

$$\widehat{Z}_0 = \lim_{t \to 0+} \widehat{Z}_{-t} = 0$$

and thus $\limsup_{t\to 0+} \widehat{Z}_{-t} = 0$. This means that

$$\mathbf{P}\left(\widehat{Z}_{-t} > \varepsilon \text{ i. o.}\right) = 0.$$

If

$$\sum_{n=1}^{\infty} \mathbf{P}\left(Z^{(o)}_{\frac{1}{n}} > \varepsilon\right) < \infty,$$

then the Borel-Cantelli Lemma implies that there exists $\varepsilon > 0$ such that

$$\mathbf{P}\left(Z^{(o)}_{\frac{1}{n}} > \varepsilon \text{ i. o.}\right) = 0,$$

which means that $\limsup_{t\to 0+} Z^{(o)}_t = 0$ almost surely.

We manipulate the sum of probabilities as follows

$$\begin{aligned}
\sum_{n=1}^{\infty} \mathbf{P}\left(Z^{(o)}_{\frac{1}{n}} > \varepsilon\right) &= \sum_{n=1}^{\infty} \mathbf{P}\left(\left(\frac{1}{n}\right)^{2H} \widehat{Z}_{-n} > \varepsilon\right) \\
&= \sum_{n=1}^{\infty} \mathbf{P}\left(\left(\frac{1}{n}\right)^{2H} n^H \widehat{Z}_{-1} > \varepsilon\right) \\
&= \sum_{n=1}^{\infty} \mathbf{P}\left(\left(\frac{1}{n}\right)^{H} \widehat{Z}_{1} > \varepsilon\right) \\
&= \sum_{n=1}^{\infty} \mathbf{P}\left(\widehat{Z}_{1} > \varepsilon n^H\right).
\end{aligned}$$

For the term inside the summation on the right-hand side, we have the upper bound

$$\mathbf{P}(\widehat{Z}_1 > \varepsilon n^H) \leq \frac{\mathbf{E}(\widehat{Z}_1^{2k})}{\varepsilon^{2k} n^{2kH}}, \tag{1.19}$$

which is obtained using inequality

$$\begin{aligned}
\mathbf{E}(\widehat{Z}_1^{2k}) &\geq \mathbf{E}(\widehat{Z}_1^{2k} 1_{\{\widehat{Z}_1 > \varepsilon n^H\}}) \\
&\geq \left(\varepsilon n^H\right)^{2k} \mathbf{P}(\widehat{Z}_1 > \varepsilon n^H).
\end{aligned}$$

Using (1.19), we infer

$$\begin{aligned}
\sum_{n=1}^{\infty} \mathbf{P}\left(Z^{(o)}_{\frac{1}{n}} > \varepsilon\right) &= \sum_{n=1}^{\infty} \mathbf{P}\left(\widehat{Z}_{1} > \varepsilon n^H\right) \\
&\leq \sum_{n=1}^{\infty} \frac{\mathbf{E}\left(\widehat{Z}_1^{2k}\right)}{\varepsilon^{2k} n^{2kH}}.
\end{aligned}$$

This sum converges for every $H > 0$, since we can choose $2k$, such that $2kH > 1$, and

$$\sum_{n=1}^{\infty} \frac{1}{n^{2kH}} < \infty.$$

Hence

$$\sum_{n=1}^{\infty} \mathbf{P}\left(Z^{(o)}_{\frac{1}{n}} > \varepsilon\right) < \infty.$$

Applying Borel-Cantelli Lemma there exists $\varepsilon > 0$ such, that

$$\mathbf{P}\left(Z^{(o)}_{\frac{1}{n}} > \varepsilon \text{ i. o.}\right) = 0$$

and therefore we deduce that $\limsup_{t\to 0+} Z^{(o)}_t = 0$ a.s. for any $Z^{(o)}$.

Next we prove that $\liminf_{t\to 0+} Z^{(o)}_t = 0$ a.s.. This follows from the fact that $\{-Z^{(o)}_t, t \geq 0\}$ is also fractional Brownian motion and, hence

$$\liminf_{t\to 0+} Z^{(o)}_t = -\limsup_{t\to 0+}\left(-Z^{(o)}_t\right) = 0 \text{ a.s.}.$$

Therefore the property

$$\lim_{t\to 0+} Z^{(o)}_t = \lim_{t\to 0+} \widehat{Z}_{-t} = 0 \text{ a.s.} \tag{1.20}$$

holds. We still have to prove that the integral on the right-hand side of (1.18) exists when s tends to $-\infty$. First we show that

$$\left|\int_{-\infty}^{N} \alpha e^{\alpha u}\widehat{Z}_u du\right| < \infty,$$

for small negative N. To see this, we deduce from (1.20) that

$$\lim_{s\to -\infty} \frac{\widehat{Z}_s}{|s|^{2H}} = \lim_{t\to 0+} t^{2H}\widehat{Z}_{-\frac{1}{t}} = \lim_{t\to 0+} Z^{(o)}_t = 0, \tag{1.21}$$

and therefore for any $\varepsilon > 0$ there exists $N < 0$ such that $\left|\widehat{Z}_s/s^{2H}\right| < \varepsilon$ for any $s < N$. Thus, we infer

$$\begin{aligned}\left|\int_{-\infty}^{N} \alpha e^{\alpha u}\widehat{Z}_u du\right| &= \left|\int_{-\infty}^{N} \alpha e^{\alpha u}|u|^{2H}\frac{\widehat{Z}_u}{u^{2H}} du\right| \\ &\leq \varepsilon\left|\int_{-\infty}^{N} \alpha e^{\alpha u}|u|^{2H} du\right| \\ &< \infty.\end{aligned}$$

In the case $\alpha > 0$, we can use the same argument (1.21) to prove that $e^{\alpha s}\widehat{Z}_s$ tends to zero, when s tends to $-\infty$

$$\lim_{s\to -\infty} e^{\alpha s}\widehat{Z}_s = \lim_{s\to -\infty} e^{\alpha s}|s|^{2H}\frac{\widehat{Z}_s}{|s|^{2H}} = 0,$$

thereby completing the proof. □

Remark 1.40.
1) Note that for Brownian motion ($H = \frac{1}{2}$) the result

$$\lim_{s \to -\infty} \frac{\widehat{Z}_s}{|s|^{2H}} = 0 \tag{1.22}$$

is an application of the Strong Law of Large Numbers (SLLN), that is

$$\frac{1}{n} \sum_{i=0}^{n} \left(B_{n-i} - B_{n-(i+1)} \right) = \frac{B_n}{n} \to \mathbf{E}(B_1) = 0.$$

Hence the result (1.22) appear to be a kind of strong law of large numbers for fractional Brownian motion. Note that we cannot use the Kolmogorov strong law of large numbers for fractional Brownian motion, since the Kolmogorov strong law of large numbers is valid only for independent and identically distributed random variables.

2) A different proof that ξ is well defined can be found in Maslowski and Schmalfuss [39] and in Garritdo-Atienza, Kloeden and Neuenkirch [21].

We apply the previous result, setting $x = \xi$ in (1.15) we get

$$U_t^{(Z,\alpha)} = \mathrm{e}^{-\alpha t} \int_{-\infty}^{t} \mathrm{e}^{\alpha s} d\widehat{Z}_s. \tag{1.23}$$

Theorem 1.41. *The process* $\{U_t^{(Z,\alpha)} : t \in \mathbb{R}\}$ *is stationary.*

Proof. In the case of Gaussian process by Theorem 1.3 stationarity follows from equality

$$\mathbf{E}\left(U_{t_1+h}^{(Z,\alpha)} U_{t_2+h}^{(Z,\alpha)} \right) = \mathbf{E}\left(U_{t_1}^{(Z,\alpha)} U_{t_2}^{(Z,\alpha)} \right),$$

for any $h \in \mathbb{R}$. Let $h \in \mathbb{R}$, then changing variables $r := s - h$ in both integrals, we obtain

$$\begin{aligned}
&\mathbf{E}\left(U_{t_1+h}^{(Z,\alpha)} U_{t_2+h}^{(Z,\alpha)} \right) \\
&= \mathbf{E}\left(\left(\mathrm{e}^{-\alpha(t_1+h)} \int_{-\infty}^{t_1+h} \mathrm{e}^{\alpha s} d\widehat{Z}_s \right) \left(\mathrm{e}^{-\alpha(t_2+h)} \int_{-\infty}^{t_2+h} \mathrm{e}^{\alpha s} d\widehat{Z}_s \right) \right) \\
&= \mathbf{E}\left(\left(\mathrm{e}^{-\alpha(t_1+h)} \int_{-\infty}^{t_1} \mathrm{e}^{\alpha(r+h)} d\widehat{Z}_{r+h} \right) \left(\mathrm{e}^{-\alpha(t_2+h)} \int_{-\infty}^{t_2} \mathrm{e}^{\alpha(r+h)} d\widehat{Z}_{r+h} \right) \right).
\end{aligned}$$

Since the increments of fractional Brownian motion are stationary, that is

$$\{Z_{t+h} - Z_{s+h} : s, t \in \mathbb{R}\} \overset{d}{=} \{Z_t - Z_s : s, t \in \mathbb{R}\}$$

and the integrals are pathwise Riemann-Stieltjes integrals, which actually consist of the sums of the increments of Z, we conclude

$$\begin{aligned}
&\mathbf{E}\left(U_{t_1+h}^{(Z,\alpha)} U_{t_2+h}^{(Z,\alpha)} \right) \\
&= \mathbf{E}\left(\left(\mathrm{e}^{-\alpha t_1} \int_{-\infty}^{t_1} \mathrm{e}^{\alpha r} d\widehat{Z}_r \right) \left(\mathrm{e}^{-\alpha t_2} \int_{-\infty}^{t_2} \mathrm{e}^{\alpha r} d\widehat{Z}_r \right) \right) \\
&= \mathbf{E}\left(U_{t_1}^{(Z,\alpha)} U_{t_2}^{(Z,\alpha)} \right),
\end{aligned}$$

thereby completing the proof. □

Definition 1.42. The process $U^{(Z,\alpha)} = \{U_t^{(Z,\alpha)} : t \in \mathbb{R}\}$ introduced in (1.23) is called the *stationary fractional Ornstein – Uhlenbeck process of the first kind*, abbreviated fOU(1).

1.4.4 The Doob transformation of fBm

In Brownian motion case, there are two ways to construct stationary Ornstein – Uhlenbeck processes: one via Definition 1.22 and the other via (1.4). We are interested in studying the same kind of situation for fractional Brownian motion. We will prove that for fractional Brownian motion these two processes have different finite dimensional distributions.

The time and space scaling fractional Ornstein – Uhlenbeck process is called a Doob (as also in the OU process case) transformation of fractional Brownian motion, see [16, Eq. (1.2.1)] or a Lamperti transformation of fBm, as in [12].

The Doob transformation is a deterministic time and space transformation, but a Lamperti transformation is a larger class of stochastic semi-stable processes, therefore we do not use the term the Lamperti transformation for this process. The concept semi-stable means the same as self-similarity in 1.16, self-similarity being nowadays more commonly used. See, for example, Chaumont, Panti and Rivero [10].

With Ornstein – Uhlenbeck processes we did not use the concept of Lamperti transformation, since, in addition to the above, Ornstein – Uhlenbeck processes were described in the middle of the 20th century, but the concept of Lamperti transformation of fBm did not appear until in the latter part of the 20th century. In fact the Lamperti transformation is more than mere the time scaling of a process.

Definition 1.43. Let $Z = \{Z_t : t \geq 0\}$ be the fractional Brownian motion. Then the process $X^{(D,\alpha)} = \{X_t^{(D,\alpha)} : t \in \mathbb{R}\}$ is *the Doob transformation of fBm* (abbreviated fOU) if

$$X_t^{(D,\alpha)} := \mathrm{e}^{-\alpha t} Z_{\tau_t},$$

where $t \in \mathbb{R}$, $H \in (0,1)$, $\alpha > 0$ and $\tau_t = \dfrac{H e^{\frac{\alpha}{H} t}}{\alpha}$.

2 Covariances and spectral density functions

2.1 Covariances and stationarity

2.1.1 Covariance of the Doob transformation of fBm

The covariance of the Doob transformation $X^{(D,\alpha)}$ of fBm may be directly computed from the covariance of fractional Brownian motion. Using the self-similarity of fBm, we infer that the random variable $X_t^{(D,\alpha)}$ is normally distributed with zero mean and the variance $\left(\frac{H}{\alpha}\right)^{2H}$ for all t. Calculations of the covariance concurrently imply stationarity of the process.

Proposition 2.1. *The covariance of the Doob transformation of fBm is*

$$\begin{aligned} &\mathbf{E}\left(X_t^{(D,\alpha)} X_s^{(D,\alpha)}\right) &(2.1)\\ &\quad= \frac{1}{2}\left(\frac{H}{\alpha}\right)^{2H}\left(\mathrm{e}^{\alpha(t-s)} + \mathrm{e}^{-\alpha(t-s)} - \mathrm{e}^{\alpha(t-s)}\left(1-\mathrm{e}^{-\frac{\alpha(t-s)}{H}}\right)^{2H}\right), \end{aligned}$$

when $t > s$, and therefore the Doob transformation of fBm is stationary.

Proof. Using covariance of fBm with $\tau_t = \frac{H\mathrm{e}^{\frac{\alpha}{H}t}}{\alpha}$, we calculate the covariance as follows

$$\begin{aligned} &\mathbf{E}\left(X_t^{(D,\alpha)} X_s^{(D,\alpha)}\right) &(2.2)\\ &\quad= \mathbf{E}\left(\mathrm{e}^{-\alpha t} Z_{\tau_t} \mathrm{e}^{-\alpha s} Z_{\tau_s}\right)\\ &\quad= \mathrm{e}^{-\alpha(t+s)}\frac{1}{2}\left(\frac{H}{\alpha}\right)^{2H}\left(\mathrm{e}^{2\alpha t} + \mathrm{e}^{2\alpha s} - \mathrm{e}^{2\alpha t}\left(1-\mathrm{e}^{\frac{-\alpha(t-s)}{H}}\right)^{2H}\right) &(2.3)\\ &\quad= \frac{1}{2}\left(\frac{H}{\alpha}\right)^{2H}\left(\mathrm{e}^{\alpha(t-s)} + \mathrm{e}^{-\alpha(t-s)} - \mathrm{e}^{\alpha(t-s)}\left(1-\mathrm{e}^{\frac{-\alpha(t-s)}{H}}\right)^{2H}\right). \end{aligned}$$

Since the process $X^{(D,\alpha)}$ is Gaussian and, its covariance function depends only on the difference, whence the process is stationary. □

2.1.2 Covariance of fOU(1)

The covariance function of fOU(1) is more complicated to calculate. The asymptotic formula of the covariance of $U^{(Z,\alpha)}$ is given by [12, Theorem 2.3.].

Proposition 2.2 ([12], Th. 2.3.)**.** *Let $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ and $N = 1, 2, \ldots$. Then*

$$\mathbf{E}\left(U_s^{(Z,\alpha)} U_{s+t}^{(Z,\alpha)}\right) = \frac{1}{2} \sum_{n=1}^{N} \alpha^{-2n} \left(\prod_{k=0}^{2n-1} (2H - k) \right) t^{2H-2n} + \mathbf{O}(t^{2H-2N-2}), \tag{2.4}$$

for $s \in \mathbb{R}$ and when $t \to \infty$.

2.2 On asymptotic behaviour of the Doob transformation of fBm

The OU process introduced by Definition 1.22, called the Doob transformation, is short-range dependent. Similarly OU defined in (1.4) is also short-range dependent. This follows from their covariance functions. This is one of the reasons why we consider behaviour of covariances of fOU and fOU(1), for large values of t. Hence, without a loss of generality we may assume that $t \geq 0$.

Recall that both processes $\{X_t^{(D,\alpha)} : t \in \mathbb{R}\}$ and $\{U_t^{(Z,\alpha)} : t \in \mathbb{R}\}$ are stationary.

Theorem 2.3 ([29], Prop. 3.1.)**.** *The Doob transformation of fBm $\{X_t^{(D,\alpha)} : t \geq 0\}$ is short-range dependent for all $H \in (0, 1)$.*

Proof. We start by writing the covariance of $X_t^{(D,\alpha)}$ in a more usable form. Using Taylor series

$$(1 + x)^{2H} = 1 + \sum_{n=1}^{\infty} \frac{2H(2H-1)\cdots(2H-n+1)}{n!} x^n, \text{ when } |x| < 1, \tag{2.5}$$

and setting $x = -\mathrm{e}^{-\frac{\alpha t}{H}}$, we infer

$$\begin{aligned} \left(1 - \mathrm{e}^{-\frac{\alpha t}{H}}\right)^{2H} &= 1 + \sum_{n=1}^{\infty} \frac{2H(2H-1)\cdots(2H-n+1)}{n!} \left(-\mathrm{e}^{-\frac{\alpha t}{H}}\right)^n \\ &= 1 + \sum_{n=1}^{\infty} \binom{2H}{n} \left(-\mathrm{e}^{-\frac{\alpha t}{H}}\right)^n, \end{aligned} \tag{2.6}$$

since $|\mathrm{e}^{-\frac{\alpha t}{H}}| < 1$ and $\frac{\alpha t}{H} > 0$, for $\alpha, t > 0$ and $H \in (0, 1)$.

Using Proposition 2.1 and (2.6) we compute

$$\begin{aligned} &\mathbf{E}\left(X_t^{(D,\alpha)} X_0^{(D,\alpha)}\right) \\ &= \frac{1}{2}\left(\frac{H}{\alpha}\right)^{2H} \left(\mathrm{e}^{-\alpha t} - \mathrm{e}^{\alpha t - \frac{\alpha t}{H}} \sum_{n=1}^{\infty} \binom{2H}{n} (-1)^n \mathrm{e}^{-\frac{\alpha t(n-1)}{H}} \right) \\ &= \frac{1}{2}\left(\frac{H}{\alpha}\right)^{2H} \left(\mathrm{e}^{-\alpha t} - \mathrm{e}^{-\alpha t(\frac{1}{H}-1)} \sum_{n=1}^{\infty} \binom{2H}{n} (-1)^n \mathrm{e}^{-\frac{\alpha t(n-1)}{H}} \right) \\ &= \frac{1}{2}\left(\frac{H}{\alpha}\right)^{2H} \mathrm{e}^{-\alpha t} - \frac{1}{2}\left(\frac{H}{\alpha}\right)^{2H} \mathrm{e}^{-\alpha t(\frac{1}{H}-1)} \sum_{n=1}^{\infty} \binom{2H}{n} (-1)^n \mathrm{e}^{-\frac{\alpha t(n-1)}{H}}. \end{aligned} \tag{2.7}$$

Obviously, the sum of the first part of the previous covariance, namely

$$\sum_{n=1}^{\infty} \frac{1}{2} \left(\frac{H}{\alpha}\right)^{2H} \mathrm{e}^{-\alpha n}$$

converges as the geometric series. Therefore we consider the sum of the latter part. We find an upper bound of the sum as follows

$$\begin{aligned}
\left| \sum_{n=1}^{\infty} \binom{2H}{n} (-1)^n \mathrm{e}^{-\frac{\alpha t(n-1)}{H}} \right| &= \left| \sum_{n=0}^{\infty} \binom{2H}{n+1} (-1)^{n+1} \mathrm{e}^{-\frac{\alpha t n}{H}} \right| \\
&= \left| -2H + \sum_{n=1}^{\infty} \frac{n-2H}{n+1} \binom{2H}{n} (-1)^n \mathrm{e}^{-\frac{\alpha t n}{H}} \right| \\
&\leq |-2H| + \left| \sum_{n=1}^{\infty} \binom{2H}{n} (-1)^n \mathrm{e}^{-\frac{\alpha t n}{H}} \right|, \qquad (2.8)
\end{aligned}$$

since $\left|\frac{n-2H}{n+1}\right| < 1$. It is allowed to use triangle inequality in (2.8), since the sum

$$\sum_{n=1}^{\infty} \frac{n-2H}{n+1} \binom{2H}{n} (-1)^n \mathrm{e}^{-\frac{\alpha t n}{H}}$$

is not actually alternative

- if $H < \frac{1}{2}$, the sum is always positive, because $2H$ is positive, $2H - 1$ is negative and all the rest ($n - 3$ pieces) are negative. So, if n is odd, then $n - 2$ is also odd and the sum is positive.
- if $H > \frac{1}{2}$, the sum is always negative, since $2H$ is positive, $2H - 1$ is positive and all the rest ($n - 3$ pieces) are negative. So, if n is odd, then $n - 3$ is even and the sum is negative and if n is even, then $n - 3$ is odd and the sum is negative.

Using (2.6), we obtain

$$\left| \sum_{n=1}^{\infty} \binom{2H}{n} (-1)^n \mathrm{e}^{-\frac{\alpha t(n-1)}{H}} \right| \leq |-2H| + \left| (1 - \mathrm{e}^{-\frac{\alpha t}{H}})^{2H} - 1 \right| < 4.$$

Hence

$$\left| \mathrm{e}^{-\alpha t\left(\frac{1}{H}-1\right)} \sum_{n=1}^{\infty} \binom{2H}{n} (-1)^n \mathrm{e}^{-\frac{\alpha t(n-1)}{H}} \right| < 4\mathrm{e}^{-\alpha t\left(\frac{1}{H}-1\right)}.$$

Note that $H \in (0,1)$ implies that $\frac{1}{H} - 1 > 0$.

Since $0 < \frac{1}{H} - 1 \leq 1$, for $H \geq \frac{1}{2}$, it holds

$$\left| \mathbf{E}\left(X_t^{(D,\alpha)} X_0^{(D,\alpha)} \right) \right| \leq K |\mathrm{e}^{-\alpha t(\frac{1}{H}-1))}|,$$

for some $K \in \mathbb{R}_+$ and further

$$\mathbf{E}\left(X_t^{(D,\alpha)} X_0^{(D,\alpha)} \right) = \mathbf{O}(\mathrm{e}^{-\alpha t(\frac{1}{H}-1))}) \qquad (2.9)$$

by Remark 1.17. In the case $H < \frac{1}{2}$, it holds

$$\mathbf{E}\left(X_t^{(D,\alpha)} X_0^{(D,\alpha)}\right) = \mathbf{O}(\mathrm{e}^{-\alpha t}). \tag{2.10}$$

Thus we have

$$\sum_{n=1}^{\infty} \mathbf{E}\left(X_n^{(D,\alpha)} X_0^{(D,\alpha)}\right) \leq K \sum_{n=1}^{\infty} \mathrm{e}^{-\alpha n(\frac{1}{H}-1)} < \infty \tag{2.11}$$

or

$$\sum_{n=1}^{\infty} \mathbf{E}\left(X_n^{(D,\alpha)} X_0^{(D,\alpha)}\right) \leq K \sum_{n=1}^{\infty} \mathrm{e}^{-\alpha n} < \infty. \tag{2.12}$$

Applying Definition 1.18, we note that the process $\{X_t^{(D,\alpha)} : t \geq 0\}$ is short-range dependent. □

We obtain the following corollary directly from the proof of the previous theorem.

Corollary 2.4. *For the covariance function of the Doob transformation of fBm $\{X_t^{(D,\alpha)} : t \geq 0\}$, we have*

$$\mathbf{E}\left(X_t^{(D,\alpha)} X_0^{(D,\alpha)}\right) = \mathbf{O}(\mathrm{e}^{-\alpha t}), \text{ if } H < \frac{1}{2}$$
$$\mathbf{E}\left(X_t^{(D,\alpha)} X_0^{(D,\alpha)}\right) = \mathbf{O}(\mathrm{e}^{-\alpha t(\frac{1}{H}-1)}), \text{ if } H \geq \frac{1}{2}.$$

Theorem 2.5 ([29], Prop. 2.4.)**.** *The fractional Ornstein – Uhlenbeck process of the first kind $\{U_t^{(Z,\alpha)} : t \in \mathbb{R}\}$ is long-range dependent if $H > \frac{1}{2}$, and short-range dependent if $H < \frac{1}{2}$.*

Proof. Let $c \in \mathbb{R}$ be arbitrary. Using (2.4) of Proposition 2.2 in the case $H > \frac{1}{2}$, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} |\rho_{U^{(Z,\alpha)}}(n)| &= \sum_{n=0}^{\infty} \left|\mathbf{E}\left(U_c^{(Z,\alpha)} U_{c+n}^{(Z,\alpha)}\right)\right| \\
&= \sum_{n=0}^{\infty} \left| \frac{1}{2} \sum_{k=1}^{N} \alpha^{-2k} \left(\prod_{i=0}^{2k-1} (2H-i) \right) n^{2H-2k} + \mathbf{O}(n^{2H-2N-2}) \right| \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{2} \sum_{k=1}^{N} \alpha^{-2k} \left(\prod_{i=0}^{2k-1} (2H-i) \right) n^{2H-2k} + \mathbf{O}(t^{2H-2N-2}) \right) \\
&= \frac{1}{2} \sum_{n=0}^{\infty} \alpha^{-2} 2H(2H-1) n^{2H-2} && (2.13) \\
&\quad + \frac{1}{2} \sum_{n=0}^{\infty} \sum_{k=2}^{N} \alpha^{-2k} \left(\prod_{i=0}^{2k-1} (2H-i) \right) n^{2H-2k} && (2.14) \\
&\quad + \sum_{n=0}^{\infty} \mathbf{O}(n^{2H-2N-2}). && (2.15)
\end{aligned}$$

We prove that the sums (2.14) and (2.15) are finite. First, we consider the term $\alpha^{-2k}\left(\prod_{i=0}^{2k-1}(2H-i)\right)n^{2H-2k}$ inside the sums of (2.14). Since it is positive and finite, we use the Fubini theorem (see Rudin [54, Theorem 7.8.]) to obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{k=2}^{N}\alpha^{-2k}\left(\prod_{i=0}^{2k-1}(2H-i)\right)n^{2H-2k} &= \sum_{k=2}^{N}\alpha^{-2k}\left(\prod_{i=0}^{2k-1}(2H-i)\right)\sum_{n=0}^{\infty}n^{2H-2k}\\ &< \infty,\end{aligned}$$

since $\sum_{n=0}^{\infty}n^{2H-2k}$ converges. The sum (2.15) converges, since $2H-2N-2$ is always less than -1. We know that the sum of covariances $\sum_{n=0}^{\infty}|\rho_{U^{(Z,\alpha)}(n)}|$ converges only if $\frac{1}{2}\sum_{n=0}^{\infty}\alpha^{-2}2H(2H-1)n^{2H-2}$ converges. Hence in the case $H>\frac{1}{2}$, $\sum_{n=0}^{\infty}|\rho_{U^{(Z,\alpha)}(n)}|$ diverges. Lastly, we consider the case $H<\frac{1}{2}$. In this case the sum $-\frac{1}{2}\sum_{n=0}^{\infty}\alpha^{-2}2H(2H-1)n^{2H-2}$ converges, since $H<\frac{1}{2}$ and therefore $\sum_{n=0}^{\infty}|\rho_{U^{(Z,\alpha)}(n)}|$ converges by comparison test.

Consequently the series $\sum_{n=0}^{\infty}|\rho_{U^{(Z,\alpha)}(n)}|$ converges if $H<\frac{1}{2}$ and diverges if $H>\frac{1}{2}$, thereby completing the proof. □

With the next corollary we show that the processes fOU and fOU(1) do not have the same finite dimensional distributions following the idea presented in Kaarakka [28] (see also [12]). We also present this corollary and the proof here, since it follows easily from the previous theorems.

Corollary 2.6. *The fractional Ornstein – Uhlenbeck process of first kind and the Doob transformation of fBm have not the same finite dimensional distributions.*

Proof. fOU(1) is long-range dependent if $H>\frac{1}{2}$, and short-range dependent if $H<\frac{1}{2}$ by Theorem 2.5. The Doob transformation of fBm is short-range dependent for all $H\in(0,1)$ by Theorem 2.3. These processes are Gaussian, the covariance determines its distribution uniquely and therefore their finite dimensional distributions are not the same when $H>\frac{1}{2}$. If $H<\frac{1}{2}$, both of these processes are short-range dependent, but the covariance of the Doob transformation $X^{(D,\alpha)}$ vanishes exponentially due to (2.10) and the covariance of fOU(1), $U^{(Z,\alpha)}$, vanishes as a power function on the strength of Proposition 2.2. This establishes the desired property. □

2.3 Spectral density function

Bochner [1] proved a theorem where every non-negative definite function can be represented as a Fourier transformation or Riemann – Stieltjes integral, where the integrator is an odd

[1] Salomon Bochner (1899-1982), American mathematician (born in Poland).

increasing function. Since every stationary Gaussian process has a non-negative definite covariance function (Theorem 1.4) of one variable, this theorem may be applied to find an odd increasing function Δ' that is called *spectral density function* of this process (see [18]). This function is used to find a trigonometric isometry between the Gaussian Hilbert space G (that is Hilbert space formed by Gaussian processes) and $L^2(\mathbb{R}, d\Delta)$ via the formula (from the Theorem 2.7, which we will prove later in this section)

$$\mathbf{E}(X_{t_1}X_{t_2}) = \int \mathrm{e}^{i\gamma t_1}\overline{\mathrm{e}^{i\gamma t_2}}d\Delta(\gamma), \tag{2.16}$$

for all $t_1, t_2 \in \mathbb{R}$. This is the main tool of the prediction, since it connects the two spaces and their elements.

The spectral density function is used in prediction. Dym and McKean described a prediction method in their book [18], and we give here a brief representation of this idea. Lamperti also studied the same problem in his book [36].

Let G be the Gaussian Hilbert space and let $\{X_t : t \in \mathbb{R}\} \in G$ be a random process with the spectral function Δ. Then $\{X_t : t \in \mathbb{R}\}$ is Gaussian and the conditional probability $\mathbf{P}\left(X_T \leq x \mid X_t : t \leq 0\right)$ also has Gaussian distribution. The prediction means that we solve the conditional expectation and the conditional mean square error

$$\mathbf{m}_c = \mathbf{E}\left(X_T \mid X_t : t \leq 0\right) \text{ and } \mathbf{Q}_c = \mathbf{E}\left((X_T - \mathbf{m})^2 \mid X_t : t \leq 0\right).$$

Actually the conditional expectation $\mathbf{m}_c$ is the perpendicular projection of X_T onto the space generated by $\{X_t : t \leq 0\}$. In other words, it is the best linear approximation to X_T given $\{X_t : t \leq 0\}$ and for the Gaussian case the linear approximation is the best possible. Dym and McKean describe a way to solve $\mathbf{m}_c$ and $\mathbf{Q}_c$ for stationary Gaussian processes.

We may calculate the spectral density function $\Delta'(\gamma)$ using the covariance function of X. If we have a spectral density function of a stationary Gaussian process, with some condition, then using the spectral density function we get the connection between the spaces G and $L^2(\mathbb{R}, d\Delta)$.

Let $L'_{[a,b]}$ be the closed subspace of $L^2(\mathbb{R}, d\Delta)$ spanned by $\{\mathrm{e}^{i\gamma t} : a \leq t \leq b\}$. In this particular case, the conditional expectation of X_T is actually a projection onto $L'_{(-\infty,0)}$. To calculate the projection, we apply another isometry between a Hardy class in the upper half plane H^{2+} and the space of the Fourier transformations of the space $L'_{(-\infty,0)}$. Then we do the projection using the Hardy Class and the outer function of the Hardy class. These classes and their properties are all described precisely in [18].

Prediction theory is very interesting and applicable, for example, in finance, and it might be one object of our interest in future, but it does not belong in the main focus of this dissertation and so we concentrate on isometry (2.16).

The next theorem and the sketch of its proof are from [18], where the writers state that " the proof is adapted from Carleman [1944]." In this chapter there are many integrals, where the integration is over the whole real line. For notational simplification, we do not indicate this in the proof. Instead of nondecreasing, which was used in [18], we use term increasing here meaning the same.

In the proof of the Bochner Theorem we need Fourier transformation and its inverse transformation, therefore we define these transformations: *The Fourier transformation* of a function f is

$$\wedge : \widehat{f}(\gamma) = \int f(x) \mathrm{e}^{i\gamma x} dx,$$

and its *inverse Fourier transformation* is

$$\vee : \check{f}(x) = \frac{1}{2\pi} \int f(\gamma) \mathrm{e}^{-i\gamma x} d\gamma.$$

There exists a Fourier transformation and its inverse transformation, see, for example, Kaplan [30, p. 519] if

$$\int_{-\infty}^{\infty} |f(\gamma)| d\gamma < \infty. \tag{2.17}$$

Theorem 2.7. *[The Bochner Theorem] If the covariance function* $\mathbf{Q}$ *of a stationary Gaussian process* X *is continuous at zero, then* $\mathbf{Q}$ *is possible to express as*

$$\mathbf{Q}(t) = \int \mathrm{e}^{i\gamma t} d\Delta(\gamma), \tag{2.18}$$

with an odd increasing function Δ *satisfying* $\lim\limits_{\gamma \to \infty} \Delta(\gamma) < \infty$.

The proof of the Bochner theorem is long and complicated. In many books it is written very briefly and details are omitted. For the sake of completeness we want to clarify its details.

In the Bochner theorem, the covariance is a function of t. Substituting in (2.18) $t := t_2 - t_1$, we have

$$\mathbf{Q}(t_2 - t_1) = \int \mathrm{e}^{i\gamma(t_2 - t_1)} d\Delta(\gamma) = \int \mathrm{e}^{i\gamma t_2} \overline{(\mathrm{e}^{i\gamma t_1})} d\Delta(\gamma).$$

Proof. The Bochner Theorem. We first write an outline of the proof.

- We start by defining $u(\omega) = u(a, b) = \int \mathrm{e}^{i(at + bi|t|)} \mathbf{Q}(t) dt$.
- Then we prove that in the upper half plane the function $u(\omega)$ is bounded, harmonic and non-negative
- Since $u(\omega)$ is bounded, harmonic and non-negative in the upper half plane, we are allowed to apply the Poisson formula and obtain $u(\omega) = \dfrac{b}{\pi} \displaystyle\int \frac{d\Delta(\gamma)}{(\gamma - a)^2 + b^2}$, where Δ is an odd increasing function.
- Finally we use the inverse Fourier transform to obtain the function $\mathbf{Q}(t)$.

Note that we have $\omega = a + bi$ in this proof. Consider the Fourier transformation given by

$$\begin{aligned} u(\omega) &= \left[\mathrm{e}^{-b|t|} \mathbf{Q}(t) \right]^{\wedge} (a) \\ &= \int \mathrm{e}^{iat} \mathrm{e}^{-b|t|} \mathbf{Q}(t) dt \\ &= \int \mathrm{e}^{i(at + bi|t|)} \mathbf{Q}(t) dt. \end{aligned} \tag{2.19}$$

First we prove that $u(\omega)$ is bounded and harmonic in the upper half plane.

Let $b > 0$, then

$$\begin{aligned}
|u(\omega)| &= \left|\int \mathrm{e}^{iat}\mathrm{e}^{-b|t|}\mathbf{Q}(t)dt\right| \\
&\leq \int |\mathrm{e}^{iat}|\mathrm{e}^{-b|t|}|\mathbf{Q}(t)|dt \\
&\leq \max_{t\geq 0}|\mathbf{Q}(t)|\int \mathrm{e}^{-b|t|}dt \\
&\leq \frac{2}{b}\max_{t\geq 0}|\mathbf{Q}(t)| \\
&= \frac{2}{b}\|\mathbf{Q}\|_\infty \qquad (2.20)\\
&= \frac{2}{b}\mathbf{Q}(0),
\end{aligned}$$

where we obtain the last equation using the Lemma 1.14, which says that the maximum value of $\mathbf{Q}$ is attained at zero. Hence $u(\omega)$ is bounded.

To prove that $u(\omega)$ is harmonic, we compute the Laplacian of $u(\omega)$ as follows

$$\begin{aligned}
&\frac{\partial^2}{\partial a^2}u(\omega) + \frac{\partial^2}{\partial b^2}u(\omega) \\
&\quad = \frac{\partial^2}{\partial a^2}\int \mathrm{e}^{iat}\mathrm{e}^{-b|t|}\mathbf{Q}(t)dt + \frac{\partial^2}{\partial b^2}\int \mathrm{e}^{iat}\mathrm{e}^{-b|t|}\mathbf{Q}(t)dt. \qquad (2.21)
\end{aligned}$$

We are allowed to change the order of the integration and the differentiation (or to be precise the limit)

$$\frac{\partial}{\partial x}\int f(t,x)dt$$

by using the Lebesgue Dominated Convergence Theorem (for example, in Royden [53] and more precisely in Ash [1, p. 52]), if

- the absolute value of the integrand function is integrable for any x: $\int |f(t,x)|dt < \infty$,
- the absolute value of the derivative of the integrand function is bounded by some integrable function i.e. $\left|\frac{\partial}{\partial x}f(t,x)\right| \leq N(t)$ and $\int N(t)dt < \infty$.

In this case, the integrand is bounded by an integrable function, since

$$\begin{aligned}
\left|\mathrm{e}^{iat}\mathrm{e}^{-b|t|}\mathbf{Q}(t)\right| &\leq \left|\mathrm{e}^{-b|t|}\mathbf{Q}(t)\right| \\
&\leq \left|\mathrm{e}^{-b|t|}\mathbf{Q}(0)\right|
\end{aligned}$$

and

$$\begin{aligned}
\int \left|\mathrm{e}^{-b|t|}\mathbf{Q}(0)\right| dt &= |\mathbf{Q}(0)|\int \mathrm{e}^{-b|t|}dt \\
&= \frac{2}{b}|\mathbf{Q}(0)| < \infty.
\end{aligned}$$

Since the derivative of the integrand is dominated by the integrable function as follows

$$\left|\frac{\partial}{\partial a}\mathrm{e}^{iat}\mathrm{e}^{-b|t|}\mathbf{Q}(t)\right| = \left|it\mathrm{e}^{iat}\mathrm{e}^{-b|t|}\mathbf{Q}(t)\right| \leq \left|t\mathrm{e}^{-b|t|}\mathbf{Q}(0)\right|,$$

where

$$|\mathbf{Q}(0)| \int \left(\left|t\mathrm{e}^{-b|t|}\right|\right) dt = \frac{2\,|\mathbf{Q}(0)|}{b^2} < \infty.$$

Also in the second term of (2.21), the absolute value of the derivative function of the integrand is also dominated by an integrable function

$$\left|\frac{\partial}{\partial b}\mathrm{e}^{iat}\mathrm{e}^{-b|t|}\mathbf{Q}(t)\right| = \left|(-|t|)\mathrm{e}^{iat}\mathrm{e}^{-b|t|}\mathbf{Q}(t)\right| \leq \left||t|\mathrm{e}^{-b|t|}\mathbf{Q}(0)\right|,$$

where

$$|\mathbf{Q}(0)| \int \left(\left||t|\mathrm{e}^{-b|t|}\right|\right) dt = \frac{2\,|\mathbf{Q}(0)|}{b^2} < \infty.$$

Hence we may change the order of integration and differentiation

$$\begin{aligned}
\frac{\partial^2 u(\omega)}{\partial a^2} + \frac{\partial^2 u(\omega)}{\partial b^2} &= \frac{\partial}{\partial a}\int \frac{\partial}{\partial a}\mathrm{e}^{iat}\mathrm{e}^{-b|t|}\mathbf{Q}(t)dt && (2.22)\\
&\quad + \frac{\partial}{\partial b}\int \frac{\partial}{\partial b}\mathrm{e}^{iat}\mathrm{e}^{-b|t|}\mathbf{Q}(t)dt \\
&= \frac{\partial}{\partial a}\int it\mathrm{e}^{iat}\mathrm{e}^{-b|t|}\mathbf{Q}(t)dt && (2.23)\\
&\quad + \frac{\partial}{\partial b}\int (-|t|)\mathrm{e}^{iat}\mathrm{e}^{-b|t|}\mathbf{Q}(t)dt.
\end{aligned}$$

Similarly we may change the order of the integration and the differentiation again, since

$$\left|it\mathrm{e}^{iat}\mathrm{e}^{-b|t|}\mathbf{Q}(t)\right| \leq \left|t\mathrm{e}^{-b|t|}\mathbf{Q}(0)\right|$$

and also

$$\left|(-|t|)\mathrm{e}^{iat}\mathrm{e}^{-b|t|}\mathbf{Q}(t)\right| \leq \left|t\mathrm{e}^{-b|t|}\mathbf{Q}(0)\right|,$$

where

$$\int \left|t\mathrm{e}^{-b|t|}\mathbf{Q}(0)\right| dt < \infty.$$

Thus

$$\frac{\partial}{\partial a}\left|it\mathrm{e}^{iat}\mathrm{e}^{-b|t|}\mathbf{Q}(t)\right| = \left|t^2\mathrm{e}^{iat}\mathrm{e}^{-b|t|}\mathbf{Q}(t)\right| \leq \left|t^2\mathrm{e}^{-b|t|}\mathbf{Q}(0)\right|,$$

and

$$\frac{\partial}{\partial b}\left|t\mathrm{e}^{iat}\mathrm{e}^{-b|t|}\mathbf{Q}(t)\right| = \left|t^2\mathrm{e}^{iat}\mathrm{e}^{-b|t|}\mathbf{Q}(t)\right| \leq \left|t^2\mathrm{e}^{-b|t|}\mathbf{Q}(0)\right|,$$

where

$$|\mathbf{Q}(0)| \int \left|t^2\mathrm{e}^{-b|t|}\right| = \frac{4\,|\mathbf{Q}(0)|}{b^3} < \infty.$$

Hence

$$\begin{aligned}
&\frac{\partial^2 u(\omega)}{\partial a^2}+\frac{\partial^2 u(\omega)}{\partial b^2}\\
&\qquad=\int\frac{\partial}{\partial a}it\mathrm{e}^{iat}\mathrm{e}^{-b|t|}\mathbf{Q}(t)dt+\int\frac{\partial}{\partial b}(-|t|)\mathrm{e}^{iat}\mathrm{e}^{-b|t|}\mathbf{Q}(t)dt\\
&\qquad=-\int t^2\mathrm{e}^{iat}\mathrm{e}^{-b|t|}\mathbf{Q}(t)dt+\int t^2\mathrm{e}^{iat}\mathrm{e}^{-b|t|}\mathbf{Q}(t)dt\\
&\qquad=0
\end{aligned}$$

and therefore $u(\omega)$ is harmonic.

We also prove that $u(\omega)$ is non-negative. Since

$$-\frac{1}{b}\frac{1}{2}\left(\lim_{k\to-\infty}(-1+\mathrm{e}^{bk})+\lim_{h\to\infty}(\mathrm{e}^{-bh}-1)\right)=\frac{1}{b},$$

we infer

$$\frac{1}{b}=-\frac{1}{b}\frac{1}{2}\int_{-\infty}^{\infty}(-b)\mathrm{e}^{-b|t|}dt.$$

Substituting this equation into (2.19):

$$\begin{aligned}
\frac{1}{b}u(\omega)&=\frac{1}{2}\left(-\frac{1}{b}\right)\int(-b)\mathrm{e}^{-b|t_1|}dt_1\int\mathrm{e}^{iat_2}\mathrm{e}^{-b|t_2|}\mathbf{Q}(t_2)dt_2\\
&=\frac{1}{2}\int\int\mathrm{e}^{-b|t_1|}\mathrm{e}^{iat_2}\mathrm{e}^{-b|t_2|}\mathbf{Q}(t_2)dt_1dt_2. \qquad (2.24)
\end{aligned}$$

We change the variables t_1 by t_1+t_2 and t_2 by t_1-t_2 and we obtain further

$$\frac{1}{b}u(\omega)=\int\int\mathrm{e}^{ia(t_1-t_2)}\mathrm{e}^{-b(|t_1+t_2|+|t_1-t_2|)}\mathbf{Q}(t_1-t_2)dt_1dt_2.$$

Since

$$|t_1+t_2|+|t_1-t_2|=|t_1|+|t_2|+||t_1|-|t_2|| \qquad (2.25)$$

for all t_1, t_2 and

$$\mathrm{e}^{-b|t|}=\frac{b}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{i\gamma t}}{\gamma^2+b^2}d\gamma, \qquad (2.26)$$

(see, for example, in Erdélyi (editor) [20, p 118]) we obtain

$$\begin{aligned}
\frac{1}{b}u(\omega)&=\int\int\mathrm{e}^{iat_1}\mathrm{e}^{-b|t_1|}\mathrm{e}^{-iat_2}\mathrm{e}^{-b|t_2|}\\
&\qquad\qquad\cdot\frac{b}{\pi}\int\frac{\mathrm{e}^{i\gamma(|t_1|-|t_2|)}}{\gamma^2+b^2}d\gamma\mathbf{Q}(t_1-t_2)dt_1dt_2\\
&=\frac{b}{\pi}\int\int\int\mathrm{e}^{iat_1-b|t_1|+i\gamma|t_1|}\mathrm{e}^{-iat_2-b|t_2|-i\gamma|t_2|}\\
&\qquad\qquad\cdot\frac{1}{\gamma^2+b^2}\mathbf{Q}(t_1-t_2)dt_1dt_2d\gamma\\
&=\frac{b}{\pi}\int\frac{1}{\gamma^2+b^2}\int\int f_\gamma(t_1)\overline{f_\gamma(t_2)}\mathbf{Q}(t_1-t_2)dt_1dt_2d\gamma,
\end{aligned}$$

where $f_\gamma(t_1) = \mathrm{e}^{-b|t|+i(at+\gamma|t|)}$ and $\overline{f_\gamma(t_2)}$ is its complex conjugate.

We may approximate the double integral by a limit of the sums as follows

$$\begin{aligned}
&\int\int f_\gamma(t_1)\overline{f_\gamma(t_2)}\mathbf{Q}(t_1-t_2)dt_1dt_2 \\
&= \lim_{n\to\infty}\sum_{j=-n}^{n}\sum_{k=-n}^{n} f_\gamma\left(\frac{j}{n}\right)\overline{f_\gamma\left(\frac{k}{n}\right)}\mathbf{Q}\left(\frac{j-k}{n}\right)\frac{2}{n}\frac{2}{n} \\
&= \lim_{n\to\infty}\sum_{j=-n}^{n}\sum_{k=-n}^{n} f_\gamma\left(\frac{j}{n}\right)\overline{f_\gamma\left(\frac{k}{n}\right)}\mathbf{E}\left(X_{\frac{j}{n}}X_{\frac{k}{n}}\right)\frac{2}{n}\frac{2}{n} \\
&= \lim_{n\to\infty}\mathbf{E}\left[\sum_{j=-n}^{n} f_\gamma\left(\frac{j}{n}\right)X_{\frac{j}{n}}\frac{2}{n}\sum_{k=-n}^{n}\overline{f_\gamma\left(\frac{k}{n}\right)}X_{\frac{k}{n}}\frac{2}{n}\right] \\
&= \lim_{n\to\infty}\mathbf{E}\left[\left|\sum_{j=-n}^{n} f_\gamma\left(\frac{j}{n}\right)X_{\frac{j}{n}}\frac{2}{n}\right|^2\right] \\
&\geq 0.
\end{aligned}$$

Consequently the function u is non-negative.

Every positive harmonic function on the upper half plane has the Poisson formula (see, for example, [18, Section 1.2])

$$u(\omega) = kb + \frac{b}{\pi}\int \frac{1}{|\gamma-\omega|^2}d\Delta(\gamma), \tag{2.27}$$

where $k \geq 0$ is a constant and Δ is an increasing function with

$$\int \frac{1}{\gamma^2+1}d\Delta(\gamma) < \infty. \tag{2.28}$$

The right-hand side of (2.27) is unbounded if $k \neq 0$. Since u is bounded k must be equal to zero for all $b \in \mathbb{R}_+$. Hence we obtain

$$u(\omega) = \frac{b}{\pi}\int \frac{d\Delta(\gamma)}{(\gamma-a)^2+b^2}. \tag{2.29}$$

Formula (2.29) is valid for increasing function Δ, which is integrable in the sense as inequality (2.28).

To prove $\Delta(\gamma)$ odd, we consider the function $u_b := u(a,b)$, for $b > 0$

$$u_b(a) = \int_{-\infty}^{\infty} \mathrm{e}^{i(at+ib|t|)}\mathbf{Q}(t)dt.$$

Since $\mathbf{Q}$ is even by Corollary 1.12, substituting $t = -s$ we obtain

$$\begin{aligned}
u_b(a) &= -\int_{\infty}^{-\infty} \mathrm{e}^{i(-as+ib|-s|)}\mathbf{Q}(-s)ds, \\
&= \int_{-\infty}^{\infty} \mathrm{e}^{i(-as+ib|s|)}\mathbf{Q}(s)ds \\
&= u_b(-a),
\end{aligned}$$

hence u_b is even. Applying the Poisson formula (2.29), we note that $\Delta(\gamma)$ must be odd. The condition $\lim\limits_{\gamma\to\infty}\Delta(\gamma)<\infty$ is valid, since

$$\begin{aligned}
2\left(\Delta(N)-0\right) &= 2\int\limits_0^N d\Delta(\gamma)\\
&= 2\int\limits_0^N \lim_{b\to\infty}\frac{b^2}{\gamma^2+b^2}d\Delta(\gamma).
\end{aligned}$$

Changing the order of integration and limits using the Lebesgue Convergence Theorem [53, Ch. 11 Sec.3 Theorem 16] and applying the formula (2.20) we obtain further

$$\begin{aligned}
2\left(\Delta(N)-0\right) &\le \lim_{b\to\infty} 2\pi b\left(\frac{b}{\pi}\int\limits_0^\infty \frac{1}{\gamma^2+b^2}d\Delta(\gamma)\right)\\
&= \lim_{b\to\infty}\pi b u(0,b)\\
&\le \pi\|\mathbf{Q}\|_\infty<\infty.
\end{aligned}$$

Finally, we use the inverse formula of $u(a,b)=\left[\mathrm{e}^{-b|t|}\mathbf{Q}(t)\right]^\wedge$. The constant b is fixed and we make the transformation with respect to a. Note that in the following calculation the equality signs are almost surely (almost everywhere) valid:

$$\begin{aligned}
A &:= \mathrm{e}^{-b|t|}\mathbf{Q}(t)\\
&= \left[\mathrm{e}^{-b|t|}\mathbf{Q}(t)\right]^{\wedge\vee}=u(a,b)^\vee\\
&= \left[\frac{b}{\pi}\int\limits_\infty^\infty\frac{d\Delta(\gamma)}{(\gamma-a)^2+b^2}\right]^\vee\\
&= -\frac{1}{2\pi}\int\limits_{-\infty}^\infty\frac{b}{\pi}\int\limits_\infty^\infty \mathrm{e}^{-iat}\frac{d\Delta(\gamma)}{(\gamma-a)^2+b^2}da.
\end{aligned}$$

If we make the substitution $s=\gamma-a$ and then change the order of integrations using the Fubini Theorem in [54, Theorem 7.8.], we obtain

$$A = \frac{1}{2\pi}\int\limits_{-\infty}^\infty\left(\frac{b}{\pi}\int\limits_\infty^\infty\frac{\mathrm{e}^{ist}}{s^2+b^2}ds\right)\mathrm{e}^{-i\gamma t}d\Delta(\gamma).$$

Applying (2.26) we compute further

$$\begin{aligned}
A &= \frac{1}{2\pi}\int\limits_{-\infty}^\infty \mathrm{e}^{-b|t|}\mathrm{e}^{-i\gamma t}d\Delta(\gamma)\\
&= \mathrm{e}^{-b|t|}\frac{1}{2\pi}\int\limits_{-\infty}^\infty \mathrm{e}^{-i\gamma t}d\Delta(\gamma).
\end{aligned}$$

Now we have proved that

$$\mathbf{Q}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-i\gamma t} d\Delta(\gamma) \text{ a.s., for all } t. \tag{2.30}$$

Since $\mathbf{Q}$ is continuous and the right-hand side of (2.30) is a continuous function of t, we infer that equality (2.30) holds everywhere. An adjustment of the minus γ and 2π completes this proof as follows

$$\begin{aligned}
\mathbf{Q}(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-i\gamma t} d\Delta(\gamma) \\
&= \frac{1}{2\pi} \int_{\infty}^{-\infty} \mathrm{e}^{-i(-\gamma)t} d\Delta(-\gamma) \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{i\gamma t} d\Delta(\gamma)
\end{aligned}$$

if we embed $\frac{1}{2\pi}$ into function Δ, we obtain the assertion. □

It is possible to do Lebesgue decomposition of the measure induced by Δ denoted by $d\Delta(\gamma)$, as follows

$$d\Delta(\gamma) = d\Delta^{\circ}(\gamma) + \Delta'(\gamma)d\gamma.$$

See, for example, [53, ch.11. Sec. 6. Prop. 24.],

The singular part $d\Delta^{\circ}(\gamma)$ is singular with respect to Lebesgue measure $d\gamma$ and *the nonsingular part*, which is *absolutely continuous* with respect to the Lebesgue measure $d\gamma$.

We recall that

- $\Delta'(\gamma)d\gamma$ is absolutely continuous with respect to $d\gamma$, that is, if $\int_A d\gamma = 0$, then $\int_A \Delta'(\gamma)d\gamma = 0$, for all Lebesgue measurable A.
- $d\Delta^{\circ}(\gamma)$ is singular with respect to $d\gamma$, that is, there exists Lebesgue measurable E such that $\int_E d\gamma = 0$ and $\int_{\Omega\setminus E} d\Delta^{\circ}(\gamma) = 0$.

We have a Riemann-Stieltjes measure Δ, which can be thought of as a Borel measure. Every Borel measure can be decomposed using the Lebesgue decomposition theory as before. Hence

$$\int \mathrm{e}^{i\gamma t} d\Delta(\gamma) = \int \mathrm{e}^{i\gamma t} \Delta'(\gamma)d\gamma + \int \mathrm{e}^{i\gamma t} d\Delta^{\circ}(\gamma).$$

The function Δ' is called *the spectral density of the underlying Gaussian process.*

2.4 Spectral density functions of OU and fOU

In this section we apply the Bochner theorem to find the spectral density functions of the Ornstein-Uhlenbeck process and the Doob transformation of fBm (fOU).

2.4.1 Spectral density function of the Ornstein–Uhlenbeck process

We calculate the spectral density function of OU process. First we manipulate the covariance function and using Formula (2.26) we obtain the spectral density function. Let $t > 0$, then

$$\begin{aligned}\mathbf{Q}(t) &= \mathbf{E}(X_t X_0)\\ &= \frac{\mathrm{e}^{-\alpha t}}{2\alpha}\\ &= \frac{\mathrm{e}^{-\alpha|t|}}{2\alpha}\\ &= \frac{1}{2\alpha}\int\limits_{-\infty}^{\infty} \mathrm{e}^{i\gamma t}\frac{\alpha}{\pi}\frac{1}{\gamma^2+\alpha^2}d\gamma\\ &= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \mathrm{e}^{i\gamma t}\frac{1}{\gamma^2+\alpha^2}d\gamma.\end{aligned}$$

Hence the spectral density function of Ornstein-Uhlenbeck process is

$$\Delta'(\gamma) = \frac{1}{2\pi}\frac{1}{\gamma^2+\alpha^2}.$$

Note also that

$$\int_{-\infty}^{\infty}\left|\frac{1}{2\pi}\frac{1}{\gamma^2+\alpha^2}\right|d\gamma = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\gamma^2+\alpha^2}d\gamma = \frac{1}{2\alpha} < \infty.$$

2.4.2 Spectral density function of the Doob transformation of fBm (fOU)

Next, we derive the spectral density function of the Doob transformation of fractional Brownian motion.

Theorem 2.8. *The spectral density function of the Doob transformation of fBm (fOU) is*

$$\Delta'(\gamma) = \frac{1}{2}\left(\frac{H}{\alpha}\right)^{2H}\left(\frac{\alpha}{\pi}\frac{1}{\gamma^2+\alpha^2} - \frac{\alpha}{\pi}\sum_{n=1}^{\infty}\binom{2H}{n}\frac{(-1)^n\left(\frac{n}{H}-1\right)}{\gamma^2+\left(\frac{\alpha n}{H}-\alpha\right)^2}\right).$$

Proof.
Applying Proposition 2.1, we obtain the covariance of fOU

$$\begin{aligned}\mathbf{Q}(t) &= \mathbf{E}(X_t^{(D,\alpha)}X_0^{(D,\alpha)})\\ &= \frac{1}{2}\left(\frac{H}{\alpha}\right)^{2H}\mathrm{e}^{\alpha t}\left(1+\mathrm{e}^{-2\alpha t}\left(1-\mathrm{e}^{-\frac{\alpha t}{H}}\right)^{2H}\right).\end{aligned}$$

Using (2.7), we infer

$$\begin{aligned}\mathbf{Q}(t) &= \frac{1}{2}\left(\frac{H}{\alpha}\right)^{2H}\left(\mathrm{e}^{-\alpha t} - \mathrm{e}^{-\alpha t(\frac{1}{H}-1)}\sum_{n=1}^{\infty}\binom{2H}{n}(-1)^n\mathrm{e}^{-\frac{\alpha t(n-1)}{H}}\right) \qquad (2.31)\\ &= \frac{1}{2}\left(\frac{H}{\alpha}\right)^{2H}\left(\mathrm{e}^{-\alpha|t|} - \mathrm{e}^{-\alpha|t|(\frac{1}{H}-1)}\sum_{n=1}^{\infty}\binom{2H}{n}(-1)^n\mathrm{e}^{-\frac{\alpha|t|(n-1)}{H}}\right),\end{aligned}$$

since $t \geq 0$.

Due to Bochner Theorem (Theorem 2.7), the covariance function of the Gaussian process has the following form

$$\mathbf{Q}(t) = \int_{-\infty}^{\infty} \mathrm{e}^{i\gamma t} d\Delta(\gamma).$$

Applying (2.26), we deduce

$$\mathrm{e}^{-\alpha|t|} = \int_{-\infty}^{\infty} \mathrm{e}^{i\gamma t} \frac{\alpha}{\pi} \frac{1}{\gamma^2 + \alpha^2} d\gamma.$$

Since the Fourier transformation is additive we can substitute (2.26) into the sum in (2.31) and obtain

$$\begin{aligned} \mathbf{Q}(t) &= \frac{1}{2}\left(\frac{H}{\alpha}\right)^{2H} \left(\int_{-\infty}^{\infty} \mathrm{e}^{i\gamma t} \frac{\alpha}{\pi} \frac{1}{\gamma^2 + \alpha^2} d\gamma \right. \\ &\quad \left. - \sum_{n=1}^{\infty} \binom{2H}{n} (-1)^n \int_{-\infty}^{\infty} \mathrm{e}^{i\gamma t} \frac{\frac{\alpha n}{H} - \alpha}{\pi} \frac{1}{\gamma^2 + \left(\frac{\alpha n}{H} - \alpha\right)^2} d\gamma \right). \end{aligned}$$

Using [54, Theorem 1.38] we can change the order of the integration and the sum, if the following lemma holds

Lemma 2.9. *If $\alpha > 0$ and $H \in (0,1)$, then*

$$\sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \left| \binom{2H}{n} (-1)^n \mathrm{e}^{i\gamma t} \frac{\frac{\alpha n}{H} - \alpha}{\pi} \frac{1}{\gamma^2 + \left(\frac{\alpha n}{H} - \alpha\right)^2} \right| d\gamma < \infty. \tag{2.32}$$

Proof. Denoting $m = \frac{\alpha n}{H} - \alpha > 0$ and noting that $|\mathrm{e}^{i\gamma t}| = 1$, we compute as follows

$$\begin{aligned} &\int_{-\infty}^{\infty} \left| \binom{2H}{n} (-1)^n \mathrm{e}^{i\gamma t} \frac{\frac{\alpha n}{H} - \alpha}{\pi} \frac{1}{\gamma^2 + \left(\frac{\alpha n}{H} - \alpha\right)^2} \right| d\gamma \\ &= \int_{-\infty}^{\infty} |\mathrm{e}^{i\gamma t}| \left| \binom{2H}{n} \frac{\frac{\alpha n}{H} - \alpha}{\pi} \frac{1}{\gamma^2 + \left(\frac{\alpha n}{H} - \alpha\right)^2} \right| d\gamma \\ &\leq \int_{-\infty}^{\infty} \left| \binom{2H}{n} \frac{m}{\pi} \frac{1}{\gamma^2 + m^2} \right| d\gamma =: A_n. \end{aligned}$$

Making the change of variable $u = \frac{\gamma}{m}$ we obtain

$$\begin{aligned} A_n &= \left|\binom{2H}{n}\right| \int_{-\infty}^{\infty} \frac{m}{\pi} \left|\frac{1}{\gamma^2+m^2}\right| d\gamma \\ &= \left|\binom{2H}{n}\right| \int_{-\infty}^{\infty} \frac{1}{\pi} \left|\frac{1}{(u^2+1)}\right| du \\ &= \left|\binom{2H}{n}\right|. \end{aligned}$$

Hence, the left-hand side of (2.32) has $\sum_{n=1}^{\infty} A_n$ as an upper bound and we have

$$\begin{aligned} \sum_{n=1}^{\infty} A_n &= \sum_{n=1}^{\infty} \left|\binom{2H}{n}\right| \\ &= \sum_{n=1}^{\infty} \left|\frac{2H(2H-1)\cdots(2H-n+1)}{n!}\right| \\ &= \sum_{n=1}^{\infty} \left|-\frac{-2H(-2H+1)\cdots(-2H+n-1)}{n!}\right| \\ &= \sum_{n=1}^{\infty} \left|\frac{\Gamma(n-2H)}{\Gamma(-2H)\Gamma(n+1)}\right|. \end{aligned}$$

Using the recursion formula we may write

$$\Gamma(-2H) = \begin{cases} \frac{\Gamma(-2H+2)}{-2H(-2H+1)}, & \text{if } \frac{1}{2} < H < 1 \\ \frac{\Gamma(-2H+1)}{-2H}, & \text{if } 0 < H < \frac{1}{2}. \end{cases}$$

$$\Gamma(-2H) = \frac{\Gamma(-2H+2)}{-2H(-2H+1)}\Gamma(1-2H) = \frac{\Gamma(-2H+2)}{1-2H}.$$

By virtue of the Stirling formula (see, for example, Rudin [55, Section 8.22.]), the asymptotic behaviour of the Gamma function is

$$\Gamma(x+1) \sim x^{x+\frac{1}{2}} \mathrm{e}^{-x}\sqrt{2\pi}, \text{ when } x \to \infty.$$

In the other words

$$\lim_{x\to\infty} \frac{\Gamma(x+1)}{x^{x+\frac{1}{2}}\mathrm{e}^{-x}} = \sqrt{2\pi}.$$

Actually this means that there exist $d \in \mathbb{R}_+$ and $c \in \mathbb{R}_+$ such that

$$dx^{x+\frac{1}{2}}\mathrm{e}^{-x}\sqrt{2\pi} \le \Gamma(x+1) \le cx^{x+\frac{1}{2}}\mathrm{e}^{-x}\sqrt{2\pi},$$

when x is greater than some large N. Since

$$\Gamma(n-2H) \le c\mathrm{e}^{-(n-2H-1)}(n-2H-1)^{n-2H-\frac{1}{2}}\sqrt{2\pi}$$

and

$$\Gamma(n+1) \geq d\mathrm{e}^{-n} n^{n+\frac{1}{2}} \sqrt{2\pi} \leq c\mathrm{e}^{-n} n^{n+\frac{1}{2}} \sqrt{2\pi},$$

we obtain

$$\begin{aligned}
\frac{\Gamma(n-2H)}{\Gamma(n+1)} &\leq \frac{c\mathrm{e}^{-(n-2H-1)}(n-2H-1)^{n-2H-\frac{1}{2}}}{d\mathrm{e}^{-n} n^{n+\frac{1}{2}}} \\
&= \frac{c}{d}\mathrm{e}^{1+2H} \frac{(n-2H-1)^{n-2H-\frac{1}{2}}}{n^{n+\frac{1}{2}}} \\
&= \frac{c}{d}\mathrm{e}^{1+2H} \left(\frac{n-2H-1}{n}\right)^{n-2H-\frac{1}{2}} \left(\frac{n^{n-2H-\frac{1}{2}}}{n^{n+\frac{1}{2}}}\right) \\
&= \frac{c}{d}\mathrm{e}^{1+2H} \left(1-\frac{2H+1}{n}\right)^{n} \left(1-\frac{2H+1}{n}\right)^{-(2H+\frac{1}{2})} \left(\frac{1}{n^{1+2H}}\right),
\end{aligned}$$

where

$$\left(1-\frac{2H+1}{n}\right)^{n} \to \mathrm{e}^{-(2H+1)}$$

and

$$\left(1-\frac{2H+1}{n}\right)^{-(2H+\frac{1}{2})} \to 1.$$

Hence, the inequality $\frac{\Gamma(n-2H)}{\Gamma(n+1)} \leq \frac{C}{n^{1+2H}}$ holds, when n is large enough. Since $2H > 0$ the series $\sum_{n=1}^{\infty} \frac{1}{n^{1+2H}}$ converges and also the series

$$\sum_{n=1}^{\infty} \frac{\Gamma(n-2H)}{\Gamma(n+1)} \tag{2.33}$$

converges completing the proof of Lemma 2.9. □

We can therefore change the order of the integration and the summation, leading to

$$\begin{aligned}
\mathbf{Q}(t) &= \frac{1}{2}\left(\frac{H}{\alpha}\right)^{2H} \int_{-\infty}^{\infty} \mathrm{e}^{i\gamma t} \frac{\alpha}{\pi(\gamma^2+\alpha^2)} d\gamma \\
&\quad -\frac{1}{2}\left(\frac{H}{\alpha}\right)^{2H} \int_{-\infty}^{\infty} \mathrm{e}^{i\gamma t} \sum_{n=1}^{\infty} \binom{2H}{n} \frac{(-1)^n(\frac{\alpha n}{H}-\alpha)}{\pi\left(\gamma^2+\left(\frac{\alpha n}{H}-\alpha\right)^2\right)} d\gamma.
\end{aligned}$$

Thus the spectral density function of the Doob transformation of fBm is

$$\Delta'(\gamma) = \frac{1}{2}\left(\frac{H}{\alpha}\right)^{2H} \frac{\alpha}{\pi} \left(\frac{1}{\gamma^2+\alpha^2} - \sum_{n=1}^{\infty} \binom{2H}{n} \frac{(-1)^n\left(\frac{n}{H}-1\right)}{\gamma^2+\left(\frac{\alpha n}{H}-\alpha\right)^2}\right).$$

We recall that a function f has the Fourier transformation if

$$\int_{-\infty}^{\infty} |f(\gamma)| d\gamma < \infty.$$

In the next lemma we consider that the previous condition is true in our case.

Lemma 2.10. *Let $\Delta'(\gamma)$ be the spectral density function of the Doob transformation of fBm*

$$\Delta'(\gamma) = \frac{1}{2}\left(\frac{H}{\alpha}\right)^{2H}\frac{\alpha}{\pi}\left(\frac{1}{\gamma^2+\alpha^2} - \sum_{n=1}^{\infty}\binom{2H}{n}\frac{(-1)^n\left(\frac{n}{H}-1\right)}{\gamma^2+\left(\frac{\alpha n}{H}-\alpha\right)^2}\right),$$

then

$$\int_{-\infty}^{\infty}|\Delta'(\gamma)|d\gamma < \infty,$$

and there is a Fourier transformation of $\Delta'(\gamma)$.

Proof. We compute

$$\int_{-\infty}^{\infty}|\Delta'(\gamma)|d\gamma = \int\limits_{-\infty}^{\infty}\left|\frac{1}{2}\left(\frac{H}{\alpha}\right)^{2H}\frac{\alpha}{\pi}\left(\frac{1}{\gamma^2+\alpha^2} - \sum_{n=1}^{\infty}\binom{2H}{n}\frac{(-1)^n\left(\frac{n}{H}-1\right)}{\gamma^2+\left(\frac{\alpha n}{H}-\alpha\right)^2}\right)\right|d\gamma$$

and approximate further

$$\begin{aligned}\int_{-\infty}^{\infty}|\Delta'(\gamma)|d\gamma &\leq \frac{1}{2}\left(\frac{H}{\alpha}\right)^{2H}\frac{\alpha}{\pi}\Bigg(\int\limits_{-\infty}^{\infty}\left|\frac{1}{\gamma^2+\alpha^2}\right|d\gamma \\ &\qquad + \int\limits_{-\infty}^{\infty}\left|\sum_{n=1}^{\infty}\binom{2H}{n}\frac{\frac{n}{H}-1}{\gamma^2+\left(\frac{\alpha n}{H}-\alpha\right)^2}\right|d\gamma\Bigg) \\ &\leq \frac{1}{2}\left(\frac{H}{\alpha}\right)^{2H}\frac{\alpha}{\pi}\left(\frac{\pi}{\alpha} + \int\limits_{-\infty}^{\infty}\left|\sum_{n=1}^{\infty}\binom{2H}{n}\frac{\frac{n}{H}-1}{\gamma^2+\left(\frac{\alpha n}{H}-\alpha\right)^2}\right|d\gamma\right).\end{aligned}$$

Decomposing the sum

$$\begin{aligned}&\sum_{n=1}^{\infty}\binom{2H}{n}\frac{\frac{n}{H}-1}{\gamma^2+\left(\frac{\alpha n}{H}-\alpha\right)^2} \\ &\quad = \binom{2H}{1}\frac{\frac{1}{H}-1}{\gamma^2+\left(\frac{\alpha}{H}-\alpha\right)^2} + \sum_{n=2}^{\infty}\binom{2H}{n}\frac{\frac{n}{H}-1}{\gamma^2+\left(\frac{\alpha n}{H}-\alpha\right)^2}\end{aligned}$$

and computing $\int_{-\infty}^{\infty}\binom{2H}{1}\frac{\frac{1}{H}-1}{\gamma^2+\left(\frac{\alpha}{H}-\alpha\right)^2} = \frac{2\pi H}{\alpha}$, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty}|\Delta'(\gamma)|d\gamma &\leq \frac{1}{2}\left(\frac{H}{\alpha}\right)^{2H}\frac{\alpha}{\pi}\Bigg(\frac{\pi}{\alpha} + \frac{2\pi H}{\alpha} \\ &\qquad + \int\limits_{-\infty}^{\infty}\sum_{n=2}^{\infty}\left|\binom{2H}{n}\right|\frac{\frac{n}{H}-1}{\gamma^2+\left(\left(\frac{n}{H}-1\right)\alpha\right)^2}d\gamma\Bigg).\end{aligned}$$

Changing the order of the integration and summation we infer

$$\int_{-\infty}^{\infty} |\Delta'(\gamma)| d\gamma \quad \leq \quad \frac{1}{2}\left(\frac{H}{\alpha}\right)^{2H} \frac{\alpha}{\pi}\left(\frac{\pi}{\alpha} + \frac{2\pi H}{\alpha} + \sum_{n=2}^{\infty}\left|\binom{2H}{n}\right| \int_{-\infty}^{\infty} \frac{\frac{n}{H}-1}{\gamma^2 + \left((\frac{n}{H}-1)\alpha\right)^2} d\gamma\right),$$

since $\sum_{n=2}^{\infty}\left|\binom{2H}{n}\right|$ converges, as we proved before (2.33). Computing the integral we conclude

$$\int_{-\infty}^{\infty} |\Delta'(\gamma)| d\gamma \quad \leq \quad \frac{1}{2}\left(\frac{H}{\alpha}\right)^{2H} \frac{\alpha}{\pi}\left(\frac{\pi}{\alpha} + \frac{2\pi H}{\alpha} + \sum_{n=2}^{\infty}\left|\binom{2H}{n}\right| \frac{\pi}{\alpha}\right) < \infty,$$

completing the proof. □

Consequently the function is absolutely integrable and the spectral density function is unique. By uniqueness all requirements of spectral density functions are valid and we have proved Theorem 2.8. □

3 Fractional Ornstein–Uhlenbeck processes

3.1 fOU(2), Fractional OU process of the second kind

The fractional Ornstein−Uhlenbeck process of the first kind (fOU(1)) is stationary and, in the case $H > \frac{1}{2}$ long-range dependent since its driving process is long-range dependent. In this chapter we will study how this situation is changed if we take a process which is short-range dependent as driving process. Is it possible to represent the Doob transformation of fBm (fOU) as a solution of a stochastic differential equation? What is the nature of the driving process? If the driving process of a process X is stationary, is then the process X stationary? What is its variance? We will answer these questions in this chapter.

First, we define the driving process $Y^{(\alpha)}$ of the Doob transformation of fBm. We make some changes to obtain a slightly better behaving process $Y^{(1)}$, but it still has some properties similar to $Y^{(\alpha)}$. The two-sided process $\widehat{Y}^{(1)}$ is a driving process of a fractional OU process of the second kind, called fOU(2).

3.1.1 Definition

To explain the idea, we first look at the Ornstein-Uhlenbeck process and its representation as a stochastic differential equation. Considering the OU process of Definition 1.22 given by

$$V_t = \mathrm{e}^{-\alpha t} B_{\frac{\mathrm{e}^{2\alpha t}}{2\alpha}}.$$

Using the chain rule we infer

$$\begin{aligned} dV_t &= -\alpha \mathrm{e}^{-\alpha t} B_{\frac{\mathrm{e}^{2\alpha t}}{2\alpha}} dt + \mathrm{e}^{-\alpha t} dB_{\frac{\mathrm{e}^{2\alpha t}}{2\alpha}} \\ &= -\alpha V_t dt + \mathrm{e}^{-\alpha t} dB_{\frac{\mathrm{e}^{2\alpha t}}{2\alpha}}. \end{aligned}$$

Here $\int_0^t \mathrm{e}^{-\alpha s} dB_{\frac{\mathrm{e}^{2\alpha s}}{2\alpha}}$ is the standard Brownian motion according to the Lévy characterization theorem, which states that every centered continuous local martingale M with the quadratic variation t is Brownian motion see, for example, [32]. To see this, we look at the variance of $\int_0^t \mathrm{e}^{-\alpha t} dB_{\frac{\mathrm{e}^{2\alpha t}}{2\alpha}}$, since the quadratic variation and the variance are the same in this situation. Changing the variable $\frac{\mathrm{e}^{2\alpha t}}{2\alpha}$ by u and using the Itô formula we obtain

$$\begin{aligned}
\mathbf{E}\left(\int_0^T \mathrm{e}^{-\alpha t} dB_{\frac{\mathrm{e}^{2\alpha t}}{2\alpha}}\right)^2 &= \mathbf{E}\left(\int_{\frac{1}{2\alpha}}^{\frac{\mathrm{e}^{2\alpha T}}{2\alpha}} \frac{1}{\sqrt{2\alpha u}} dB_u\right)^2 \\
&= \int_{\frac{1}{2\alpha}}^{\frac{\mathrm{e}^{2\alpha T}}{2\alpha}} \frac{1}{2\alpha u} du = T
\end{aligned}$$

which was the assertion.

In the classical OU case we have $H = \frac{1}{2}$, to extend the construction for all $H \in (0,1)$, we introduce:

Definition 3.1. For $H \in (0,1)$ and $\alpha > 0$, we define

$$Y_t^{(\alpha)} := \int_0^t \mathrm{e}^{-\alpha s} dZ_{\tau_s} = \int_0^t \mathrm{e}^{-\alpha s} dZ_{\frac{H\mathrm{e}^{\frac{\alpha s}{H}}}{\alpha}}, \tag{3.1}$$

where $\tau_t = \dfrac{H\mathrm{e}^{\frac{\alpha t}{H}}}{\alpha}$, where the stochastic integral is the pathwise Riemann-Stieltjes integral.

The process $Y^{(\alpha)}$ may be represented as the Volterra process with respect to Brownian motion, which we shall verify later in Section 3.1.4.

Proposition 3.2. *The Doob transformation of fBm $X^{(D,\alpha)}$ is a solution of the linear stochastic differential equation*

$$dX_t^{(D,\alpha)} = -\alpha X_t^{(D,\alpha)} dt + dY_t^{(\alpha)}, \tag{3.2}$$

with the random initial value

$$X_0^{(D,\alpha)} = Z_{\frac{H}{\alpha}} \sim \mathbf{N}\left(0, \left(\frac{H}{\alpha}\right)^{2H}\right). \tag{3.3}$$

Proof. Using Definition 1.43 we get the form $X_t^{(D,\alpha)} = \mathrm{e}^{-\alpha t} Z_{\tau_t}$, where Z is fractional Brownian motion and $\tau_t = \frac{H\mathrm{e}^{\frac{\alpha}{H}t}}{\alpha}$. Hence (3.3) holds. For (3.2) consider

$$\begin{aligned}
dX_t^{(D,\alpha)} &= d\left(\mathrm{e}^{-\alpha t}\right) Z_{\tau_t} + \mathrm{e}^{-\alpha t} dZ_{\tau_t} \\
&= -\alpha \mathrm{e}^{-\alpha t} Z_{\tau_t} dt + \mathrm{e}^{-\alpha t} dZ_{\tau_t} \\
&= -\alpha X_t^{(D,\alpha)} dt + dY_t^{(\alpha)}.
\end{aligned}$$

□

The next theorem, stated in Kaarakka and Salminen [29], is the key result for defining the Langevin stochastic differential equation, where the driving process is slightly simpler than $Y^{(\alpha)}$.

Proposition 3.3 ([29], Prop. 3.2.). *The property*

$$\{\alpha^H Y^{(\alpha)}_{\frac{t}{\alpha}} : t \geq 0\} \stackrel{d}{=} \{Y^{(1)}_t : t \geq 0\}$$

holds for any $\alpha > 0$.

Proof. Using the integration by parts formula we obtain a different form of the process $Y^{(\alpha)}_t$ as follows

$$\begin{aligned} Y^{(\alpha)}_t &= \int_0^t \mathrm{e}^{-\alpha s} dZ_{\tau_s} \\ &= \mathrm{e}^{-\alpha t} Z_{\tau_t} - Z_{\frac{H}{\alpha}} + \alpha \int_0^t \mathrm{e}^{-\alpha s} Z_{\tau_s} ds. \end{aligned} \tag{3.4}$$

Introducing a new variable $p := \alpha s$ and using the self-similarity property of fBm, we infer

$$\begin{aligned} \left\{\alpha^H Y^{(\alpha)}_{\frac{t}{\alpha}} : t \geq 0\right\} &= \left\{\alpha^H \left(\mathrm{e}^{-t} Z_{\tau_{\frac{t}{\alpha}}} - Z_{\frac{H}{\alpha}} + \alpha \int_0^{\frac{t}{\alpha}} \mathrm{e}^{-\alpha s} Z_{\tau_s} ds\right) : t \geq 0\right\} \\ &= \left\{\alpha^H \left(\mathrm{e}^{-t} Z_{\frac{\mathrm{e}^{\frac{t}{\alpha}} H}{\alpha}} - Z_{\frac{H}{\alpha}} + \int_0^t \mathrm{e}^{-p} Z_{\tau_{\frac{p}{\alpha}}} dp\right) : t \geq 0\right\} \\ &\stackrel{d}{=} \left\{\alpha^H \left(\frac{1}{\alpha^H} \mathrm{e}^{-t} Z_{\mathrm{e}^{\frac{t}{\alpha}} H} - \frac{1}{\alpha^H} Z_H + \int_0^t \mathrm{e}^{-p} Z_{\tau^{(1)}_p} dp\right) : t \geq 0\right\} \\ &= \left\{\int_0^t \mathrm{e}^{-p} dZ_{\tau^{(1)}_p} : t \geq 0\right\} \\ &= \left\{Y^{(1)}_t : t \geq 0\right\}, \end{aligned}$$

where $\tau^{(1)}_t = H\mathrm{e}^{\frac{t}{H}}$, and this proves the statement. □

Proposition 3.3 is inspiring since we can define a new Langevin stochastic differential equation, where the driving process is $Y^{(1)}$. In the solution of that Langevin equation we need the extended process $\widehat{Y}^{(1)}$, which is the two-sided $Y^{(1)}$ defined similarly to the two-sided fBm in the case of fOU(1).

Definition 3.4. Let $\overline{Y}^{(1)} = \{\overline{Y}^{(1)} : t \geq 0\}$ be an independent copy of $Y^{(1)}$, starting from 0. Then for $t \in \mathbb{R}$, we define the process $\{\widehat{Y}^{(1)} : t \geq 0\}$ via

$$\widehat{Y}^{(1)}_t := \begin{cases} Y^{(1)}_t, & t \geq 0, \\ \overline{Y}^{(1)}_{-t}, & t < 0. \end{cases}$$

Proposition 3.5. *The Langevin stochastic differential equation*

$$dU_t^{(D,\gamma)} = -\gamma U_t^{(D,\gamma)} dt + dY_t^{(1)},\ \gamma > 0 \tag{3.5}$$

has the solution

$$U_t^{(D,\gamma)} = \mathrm{e}^{-\gamma t} \int_{-\infty}^{t} \mathrm{e}^{\gamma s} d\widehat{Y}_s^{(1)} = \mathrm{e}^{-\gamma t} \int_{-\infty}^{t} \mathrm{e}^{(\gamma-1)s} dZ_{\tau_s^{(1)}},\ \gamma > 0, \tag{3.6}$$

where $\tau_t^{(1)} = H\mathrm{e}^{\frac{t}{H}}$.

Proof. We can prove the existence of the integral in (3.6) in the same way as in the proof of Theorem 1.39. The most important thing is to show that

$$\int_{-\infty}^{t} \mathrm{e}^{(\gamma-1)s} dZ_{\tau_s^{(1)}}$$

is well defined.

For $\gamma \in (0,1]$ we use the integration by parts in (3.4), to obtain

$$\begin{aligned} \int_{T}^{s} \mathrm{e}^{(\gamma-1)u} dZ_{\tau_u^{(1)}} &= \mathrm{e}^{(\gamma-1)s} Z_{\tau_s^{(1)}} - \mathrm{e}^{(\gamma-1)T} Z_{\tau_T^{(1)}} \\ &\quad -(\gamma-1) \int_{T}^{s} \mathrm{e}^{(\gamma-1)u} Z_{\tau_u^{(1)}} du, \end{aligned} \tag{3.7}$$

where $\tau_u^{(1)} = H\mathrm{e}^{\frac{u}{H}}$ and $T < 0$.

Note again that fractional Brownian motion Z is locally Hölder continuous of the order β for any $\beta < H$, see Theorem 1.32. Hence, for $\beta < H$

$$|Z_s - Z_0| \leq C|s|^{\beta}.$$

For any β there exists β_0 such that $\beta < \beta_0 < H$ and

$$|Z_s - Z_0| \leq C|s|^{\beta_0-\beta+\beta},$$

which means that

$$\frac{|Z_s|}{|s|^{\beta}} \leq C|s|^{\beta_0-\beta}.$$

Hence we obtain

$$\lim_{s\to 0} \frac{|Z_s|}{|s|^{\beta}} = 0 \text{ a.s. } \forall \beta < H. \tag{3.8}$$

We consider the term $\mathrm{e}^{(\gamma-1)T} Z_{\tau_T^{(1)}}$ in (3.7), when $T \to -\infty$ as follows

$$\begin{aligned} \lim_{T\to-\infty} \mathrm{e}^{-(1-\gamma)T} Z_{\tau_T^{(1)}} &= \lim_{r\to 0} \mathrm{e}^{-(1-\gamma)H\log\left(\frac{r}{H}\right)} Z_r \\ &= \lim_{r\to 0} \left(\frac{r}{H}\right)^{-(1-\gamma)H} Z_r \\ &= \lim_{r\to 0} H^{(1-\gamma)H} \frac{Z_r}{r^{(1-\gamma)H}}. \end{aligned}$$

Since $(1-\gamma)H < H$ for $\gamma > 0$ and applying (3.8), we infer

$$\lim_{T\to-\infty} \mathrm{e}^{-(1-\gamma)T} Z_{\tau_T^{(1)}} = 0.$$

Next we prove the convergence of the integral in (3.7). Changing variables, we obtain

$$\begin{aligned}
\lim_{T\to-\infty} \left| \int_T^s \mathrm{e}^{(\gamma-1)u} Z_{\tau_u^{(1)}} du \right| &\leq \lim_{T\to-\infty} \int_T^s \mathrm{e}^{(\gamma-1)u} |Z_{\tau_u^{(1)}}| du & (3.9)\\
&= \lim_{\tau_T\to 0} \int_{\tau_T}^{\tau_s} \left(\frac{r}{H}\right)^{(\gamma-1)H} \frac{H}{r} |Z_r| dr & (3.10)\\
&= \lim_{\tau_T\to 0} \int_{\tau_T}^{\tau_s} H^{-(\gamma-1)H+1} \frac{|Z_r|}{r^{-(\gamma-1)H+1}} dr\\
&= \lim_{\tau_T\to 0} \int_{\tau_T}^{\tau_s} H^{-(\gamma-1)H+1} \frac{|Z_r|}{r^{\beta}} r^{(\gamma-1)H-1+\beta} dr\\
&\leq \lim_{\tau_T\to 0} \int_{\tau_T}^{\tau_s} H^{-(\gamma-1)H+1} \frac{|Z_r|}{r^{\beta}} r^{(\gamma-1)H-1+H} dr.
\end{aligned}$$

Since we have proved in (3.8) that $\lim_{r\to 0} \frac{|Z_r|}{r^\beta} = 0$ for any $\beta < H$ and

$$\int_0^\varepsilon r^{(\gamma-1)H-1+H} dr = \int_0^\varepsilon r^{\gamma H-1} dr = \frac{\varepsilon^{\gamma H}}{\gamma H} < \infty,$$

then

$$\lim_{T\to-\infty} \left| \int_T^s \mathrm{e}^{(\gamma-1)u} Z_{\tau_u^{(1)}} du \right| < \infty$$

for $\gamma > 0$.

It is clearer to see that the integral exists when $\gamma > 1$ than when $\gamma \in (0, 1]$, for $T < 0$, the limit $\mathrm{e}^{(\gamma-1)T} Z_{\tau_T^{(1)}} \to 0$, since $\mathrm{e}^{(\gamma-1)T} \to 0$. We also proved previously that the limit of the integral

$$\lim_{T\to-\infty} \left| \int_T^s \mathrm{e}^{(\gamma-1)u} Z_{\tau_u^{(1)}} du \right|$$

is bounded for any $\gamma > 0$, in particular $\gamma > 1$.

Therefore, we conclude that the right-hand side of (3.7) is well defined, when T tends to $-\infty$. This completes the proof. □

Now we are ready to define a new family of fractional Ornstein–Uhlenbeck processes as follows.

Definition 3.6. The process $U^{(D,\gamma)}$ introduced in (3.6) is called *the fractional Ornstein–Uhlenbeck process of the second kind, abbreviated fOU(2).*

Remark 3.7. For $\gamma = 1$, we have

$$\begin{aligned} U_t^{(D,1)} &= \mathrm{e}^{-t} \int_{-\infty}^{t} dZ_{\tau_s^{(1)}} \\ &= \mathrm{e}^{-t} \int_{-\infty}^{t} dZ_{H\mathrm{e}^{\frac{s}{H}}} \\ &= \mathrm{e}^{-t} (Z_{H\mathrm{e}^{\frac{t}{H}}} - Z_0) \end{aligned}$$

by continuity

$$\lim_{t \to -\infty} Z_{H\mathrm{e}^{\frac{t}{H}}} = Z_0 = 0.$$

Also,

$$\begin{aligned} U_t^{(D,1)} &= \mathrm{e}^{-t} Z_{H\mathrm{e}^{\frac{t}{H}}} \\ &= X_t^{(D,1)}. \end{aligned}$$

3.1.2 Simulations of fractional Ornstein–Uhlenbeck processes

We have analysed mathematically the fractional Ornstein−Uhlenbeck processes. Now we present simulations of these processes. We have done these simulations in MatLab using Cholensky decomposition medhod (see, for example, Embrechts and Maejima [19]), where we calculated values in 270 points. In Figures 3.1, 3.2, 3.3 and 3.4 there are sample paths of all three fractional Ornstein−Uhlenbeck processes with Hurst constant H values of 0.15, 0.4, 0.6 and 0.8 $\gamma = \alpha = 1$. In Figures 3.5, 3.6, 3.7 and 3.8 there are sample paths of all three fractional Ornstein−Uhlenbeck processes with Hurst constant H values of 0.4 and 0.7 with two different value of γ and α 0.5 and 2.5.

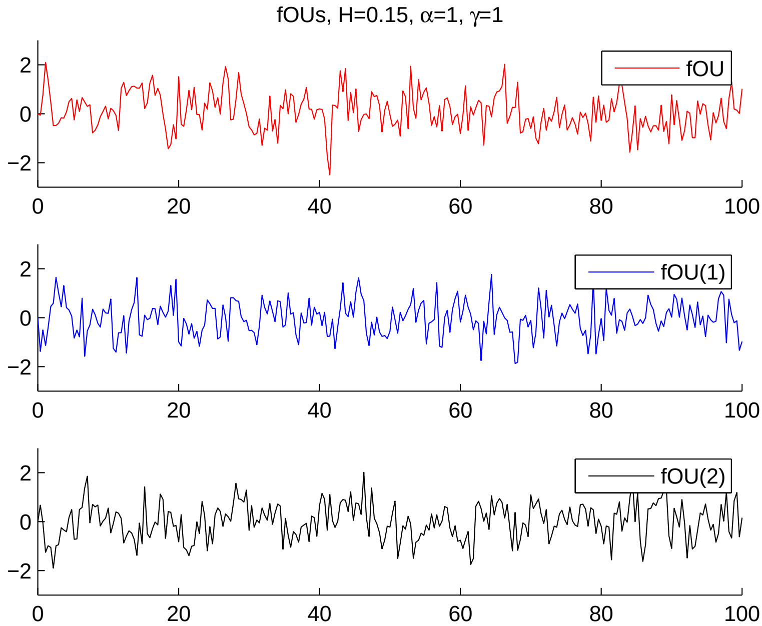

Figure 3.1: Fractional Ornstein-Uhlenbeck processes with H=0.15

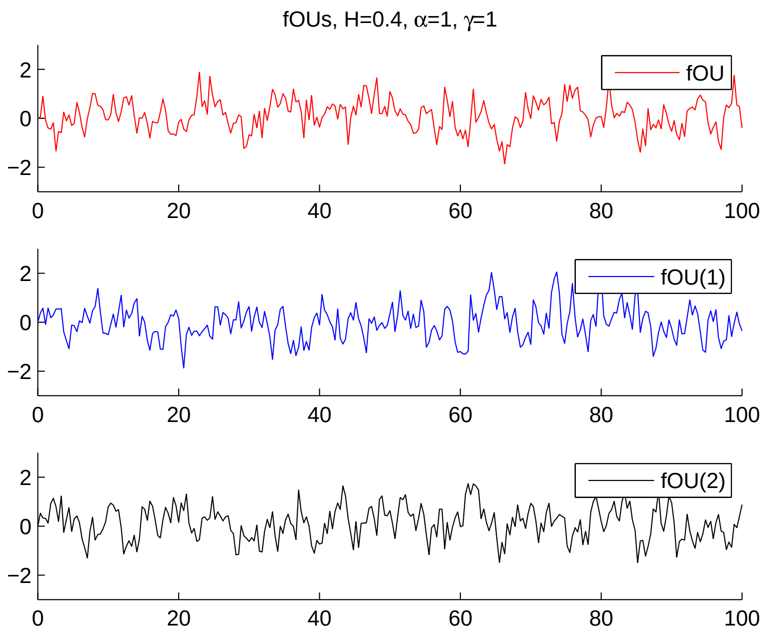

Figure 3.2: Fractional Ornstein-Uhlenbeck processes with H=0.4

In Figures 3.1 and 3.2 fractional Ornstein Uhlenbeck process of the second kind, fOU(2), and the Doob transformation of fBm, fOU, describe actually the same process, since $\alpha = \gamma = 1$. All the three processes are short-range dependent processes, when $H < \frac{1}{2}$, which is the case in these Figures 3.1 and 3.2.

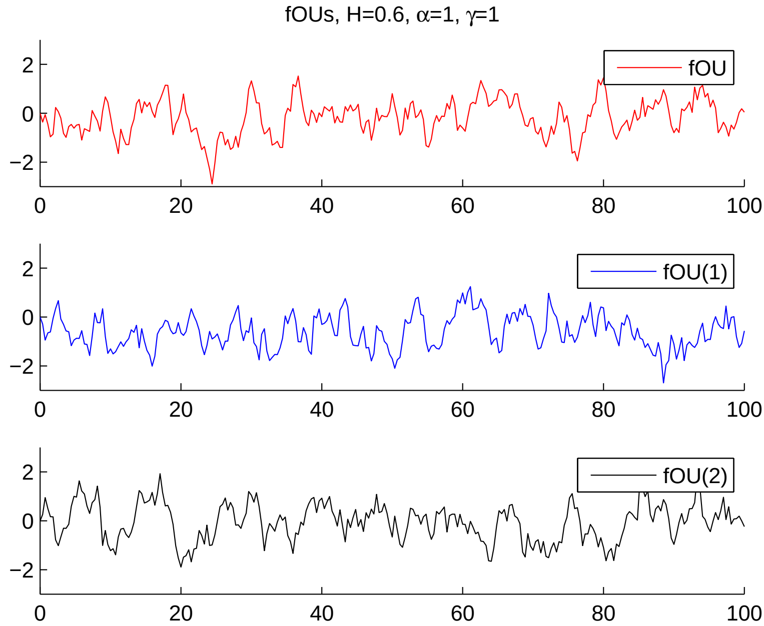

Figure 3.3: Fractional Ornstein-Uhlenbeck processes with H=0.6

In Figures 3.3 and 3.4 the fractional Ornstein Uhlenbeck processes of the second kind and the Doob transformation of fBm are both short-range dependent, but the fractional Ornstein Uhlenbeck process of the first kind, fOU(1), is long-range dependent, since $H < \frac{1}{2}$. Since $\alpha = \gamma = 1$, fOU(1) and fOU describe the same process again.

In Figures 3.5 and 3.6, we consider, how the values of the parameters γ and α impact to the sample path, when $H = 0.4$ and in figures 3.7 and 3.8 we do the same when $H = 0.7$. It is clear that the sample path is smoother with greater γ or α.

3.1.3 Stationarity of $Y^{(\alpha)}$ and $U^{(D,\gamma)}$

In this section we consider the stationarity of $Y^{(\alpha)}$ and $U^{(D,\gamma)}$. In both cases the proof of stationarity or the proof of stationarity of increments follows from the definitions of the fOU processes.

Proposition 3.8. *The process $Y^{(\alpha)}$ has stationary increments and fOU(2), that is the process $U^{(D,\gamma)}$, is stationary.*

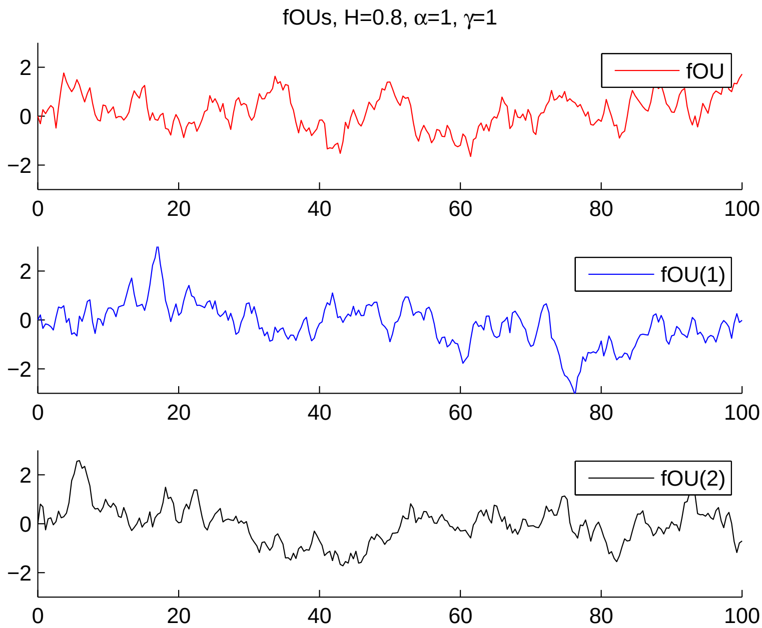

Figure 3.4: Fractional Ornstein-Uhlenbeck processes with H=0.8

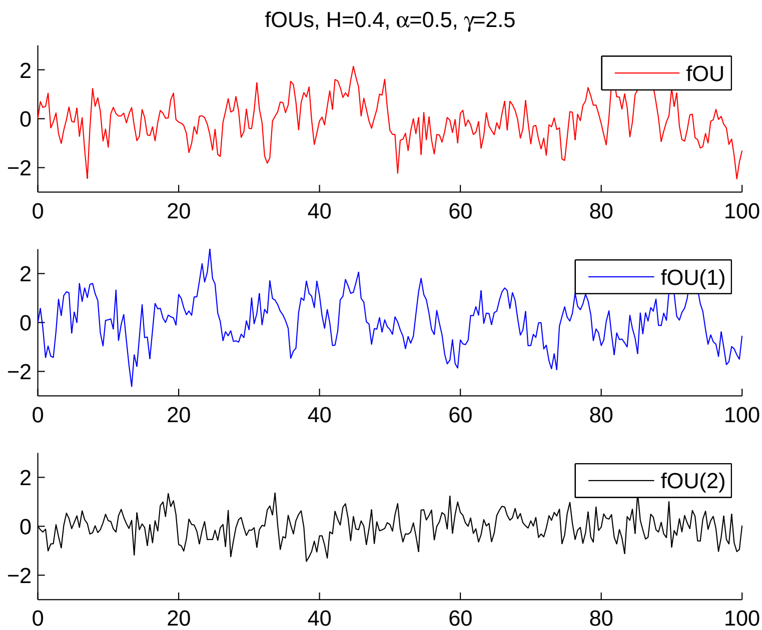

Figure 3.5: Fractional Ornstein-Uhlenbeck processes with $\gamma = 2.5$ and $\alpha = 0.5$

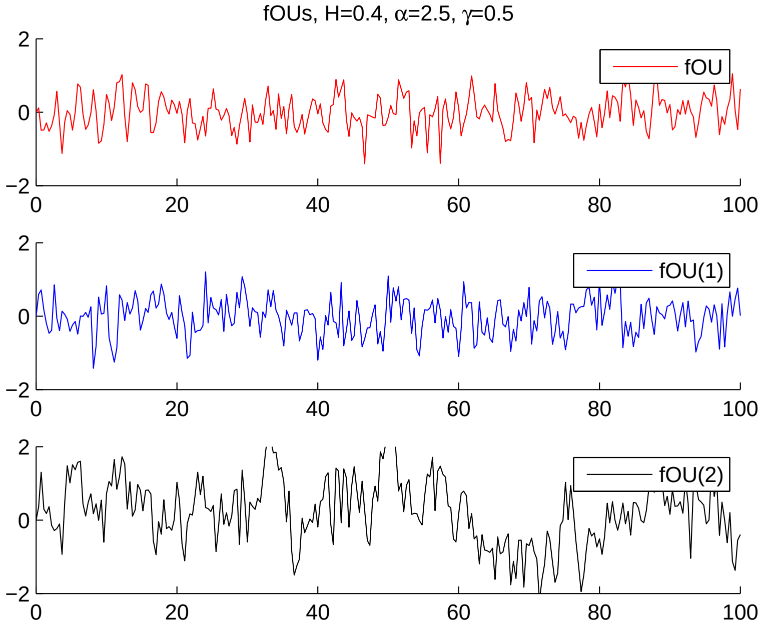

Figure 3.6: Fractional Ornstein-Uhlenbeck processes with $\gamma = 0.5$ and $\alpha = 2.5$

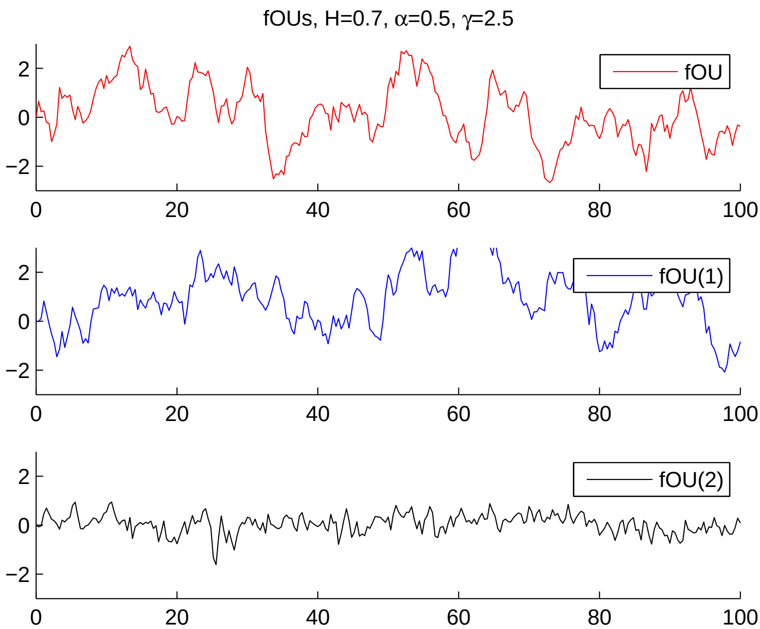

Figure 3.7: Fractional Ornstein-Uhlenbeck processes with $\gamma = 2.5$ and $\alpha = 0.5$

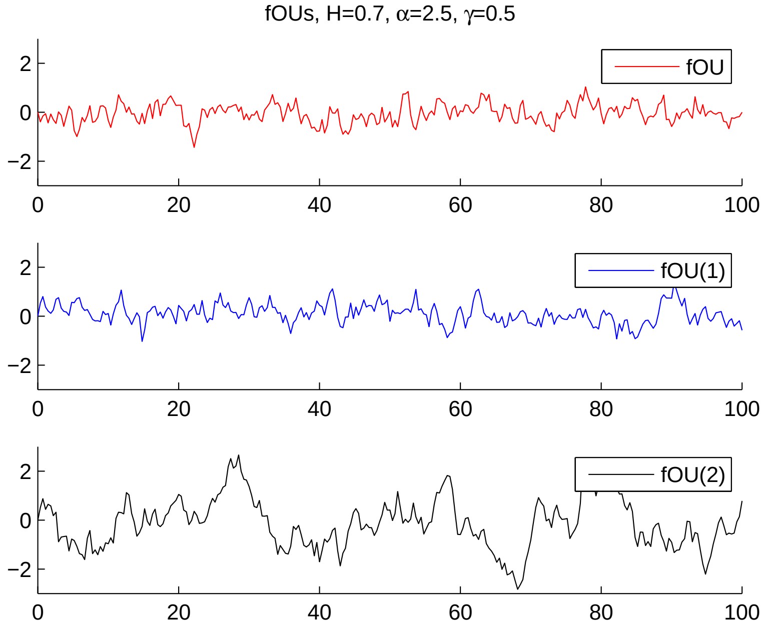

Figure 3.8: Fractional Ornstein-Uhlenbeck processes with $\gamma = 0.5$ and $\alpha = 2.5$

Proof. We use the longer form (3.4) of $Y^{(\alpha)}$, where

$$Y_{t_2}^{(\alpha)} - Y_{t_1}^{(\alpha)} = \mathrm{e}^{-\alpha t_2} Z_{\tau_{t_2}} - \mathrm{e}^{-\alpha t_1} Z_{\tau_{t_1}} + \alpha \int_{t_1}^{t_2} \mathrm{e}^{-\alpha s} Z_{\tau_s} ds$$

$$Y_{s_2}^{(\alpha)} - Y_{s_1}^{(\alpha)} = \mathrm{e}^{-\alpha s_2} Z_{\tau_{s_2}} - \mathrm{e}^{-\alpha s_1} Z_{\tau_{s_1}} + \alpha \int_{s_1}^{s_2} \mathrm{e}^{-\alpha s} Z_{\tau_s} ds,$$

and consider

$$\mathbf{E}\left(\left(Y_{t_2}^{(\alpha)} - Y_{t_1}^{(\alpha)}\right)\left(Y_{s_2}^{(\alpha)} - Y_{s_1}^{(\alpha)}\right)\right). \tag{3.11}$$

By self-similarity of fBm, we note

$$\begin{aligned}\left\{\mathrm{e}^{-\alpha(t+h)} Z_{\frac{H\mathrm{e}^{\frac{\alpha(t+h)}{H}}}{\alpha}} : t \in \mathbb{R}\right\} &= \left\{\mathrm{e}^{-\alpha(t+h)} Z_{\frac{H}{\alpha}\mathrm{e}^{\frac{\alpha t}{H}}\mathrm{e}^{\frac{\alpha h}{H}}} : t \in \mathbb{R}\right\} \\ &\overset{d}{=} \left\{\mathrm{e}^{-\alpha(t+h)} \left(\mathrm{e}^{\frac{\alpha h}{H}}\right)^H Z_{\frac{H}{\alpha}\mathrm{e}^{\frac{\alpha t}{H}}} : t \in \mathbb{R}\right\} \\ &= \left\{\mathrm{e}^{-\alpha t} Z_{\frac{H}{\alpha}\mathrm{e}^{\frac{\alpha t}{H}}} : t \in \mathbb{R}\right\}.\end{aligned} \tag{3.12}$$

Using (3.12) and changing the variable s by $u - h$, we infer further

$$\begin{aligned}&\mathbf{E}\left(\left(Y_{t_2}^{(\alpha)} - Y_{t_1}^{(\alpha)}\right)\left(Y_{s_2}^{(\alpha)} - Y_{s_1}^{(\alpha)}\right)\right) \\ &= \mathbf{E}\left(\left(Y_{t_2+h}^{(\alpha)} - Y_{t_1+h}^{(\alpha)}\right)\left(Y_{s_2+h}^{(\alpha)} - Y_{s_1+h}^{(\alpha)}\right)\right),\end{aligned}$$

where

$$Y^{(\alpha)}_{t_2+h} - Y^{(\alpha)}_{t_1+h} = \mathrm{e}^{-\alpha(t_2+h)} Z_{\tau_{t_2+h}} - \mathrm{e}^{-\alpha(t_1+h)} Z_{\tau_{t_1+h}} + \alpha \int_{t_1+h}^{t_2+h} \mathrm{e}^{-\alpha(u-h)} Z_{\tau_{u-h}} du$$

$$Y^{(\alpha)}_{s_2+h} - Y^{(\alpha)}_{s_1+h} = \mathrm{e}^{-\alpha(s_2+h)} Z_{\tau_{s_2+h}} - \mathrm{e}^{-\alpha(s_1+h)} Z_{\tau_{s_1+h}} + \alpha \int_{s_1+h}^{s_2+h} \mathrm{e}^{-\alpha(u-h)} Z_{\tau_{u-h}} du$$

and $0 < s_2 < s_2 < t_1 < t_2$ and $h > 0$. We deduce using Theorem 1.3 that the increment process of $Y^{(\alpha)}$ is stationary.

Next, we prove that the process $U^{(D,\alpha)}$ is stationary. Using Proposition 3.5 we calculate the covariance

$$\mathbf{E}\left(U^{(D,\alpha)}_t U^{(D,\alpha)}_s\right) = \mathbf{E}\left(\mathrm{e}^{-\gamma t} \int_{-\infty}^{t} \mathrm{e}^{(\gamma-1)p} dZ_{\tau^{(1)}_p} \mathrm{e}^{-\gamma s} \int_{-\infty}^{s} \mathrm{e}^{(\gamma-1)p} dZ_{\tau^{(1)}_p}\right)$$

To prove that process $U^{(D,\alpha)}$ is stationary it is enough to show that

$$\mathbf{E}\left(U^{(D,\alpha)}_{t+h} U^{(D,\alpha)}_{s+h}\right) = \mathbf{E}\left(U^{(D,\alpha)}_t U^{(D,\alpha)}_s\right).$$

When we calculate the covariance

$$\mathbf{E}\left(U^{(D,\alpha)}_{t+h} U^{(D,\alpha)}_{s+h}\right) = \mathbf{E}\left(\mathrm{e}^{-\gamma(t+h)} \int_{-\infty}^{t+h} \mathrm{e}^{(\gamma-1)p} dZ_{\tau^{(1)}_p}\, \mathrm{e}^{-\gamma(s+h)} \int_{-\infty}^{s+h} \mathrm{e}^{(\gamma-1)p} dZ_{\tau^{(1)}_p}\right)$$

changing the variable p by $r + h$, we obtain

$$\mathbf{E}\left(U^{(D,\alpha)}_{t+h} U^{(D,\alpha)}_{s+h}\right) = \mathbf{E}\left(\mathrm{e}^{-\gamma(t+h)} \int_{-\infty}^{t} \mathrm{e}^{(\gamma-1)(r+h)} dZ_{\tau^{(1)}_{r+h}}\, \mathrm{e}^{-\gamma(s+h)} \int_{-\infty}^{s} \mathrm{e}^{(\gamma-1)(r+h)} dZ_{\tau^{(1)}_{r+h}}\right).$$

Using self-similarity of fractional Brownian motion, we infer further

$$\begin{aligned}
\mathbf{E}\left(U^{(D,\alpha)}_{t+h} U^{(D,\alpha)}_{s+h}\right) &= \mathbf{E}\left(\mathrm{e}^{-\gamma(t+h)} \int_{-\infty}^{t} \mathrm{e}^{(\gamma-1)(r+h)} e^h dZ_{\tau^{(1)}_r}\, \mathrm{e}^{-\gamma(s+h)} \int_{-\infty}^{s} \mathrm{e}^{(\gamma-1)(r+h)} \mathrm{e}^h dZ_{\tau^{(1)}_r}\right) \\
&= \mathbf{E}\left(\mathrm{e}^{-\gamma t} \int_{-\infty}^{t} \mathrm{e}^{(\gamma-1)r} dZ_{\tau^{(1)}_r}\, \mathrm{e}^{-\gamma s} \int_{-\infty}^{s} \mathrm{e}^{(\gamma-1)r} \mathrm{e}^h dZ_{\tau^{(1)}_r}\right) \\
&= \mathbf{E}\left(U^{(D,\alpha)}_t U^{(D,\alpha)}_s\right),
\end{aligned}$$

thereby completing the proof.

□

3.1.4 Processes $Y^{(\alpha)}$ and $Y^{(1)}$ as stochastic integrals with respect to Brownian motion

When we present a stochastic process as an integral, where the integrator is a semimartingale and the integrand bounded deterministic real-valued function, then we are dealing with Volterra processes see, for example, Mytnic and Neuman [44]

$$M(t) = \int_0^t F(t,r)dX(r),\ t \in \mathbb{R}_+.$$

It is possible to represent the processes $Y^{(\alpha)}$ and $Y^{(1)}$ with respect to the stochastic integral of Brownian motion using the representation of fractional Brownian motion.

There are some relationships between two-sided Brownian motion and two-sided fractional Brownian motion. The first presentation is due to Mandelbrot and Van Ness [38]. Almost simultaneously Molchan and Golosov published their presentation, where the one-sided fractional Brownian motion is presented as the integral with respect to Brownian motion. Proofs of both presentations can be found, for example, in Jost [27].

We use the presentation by Norros and Virtamo [46], from which the presentation of Mandelbrot and Van Ness can actually be attained.

When $H > \frac{1}{2}$ fractional Brownian motion may be represented with respect to a stochastic integral of Brownian motion and Brownian motion may be presented with respect to the stochastic integral of fractional Brownian motion as follows

$$Z_t = C\int_0^t K(t,s)dB_s, \tag{3.13}$$

$$B_t = C'\int_0^t K_W(t,s)dZ_s, \tag{3.14}$$

where we use the abbreviation

$$\begin{aligned} K(t,s) &= \Big(H-\frac{1}{2}\Big)s^{-H+\frac{1}{2}}\int_s^t u^{H-\frac{1}{2}}(u-s)^{H-\frac{3}{2}}du, \\ K_W(t,s) &= s^{-H+\frac{1}{2}}t^{H-\frac{1}{2}}(t-s)^{-H+\frac{1}{2}} \\ &\quad -s^{-H+\frac{1}{2}}\Big(H-\frac{1}{2}\Big)\int_s^t u^{H-\frac{3}{2}}(u-s)^{-H+\frac{1}{2}}du, \end{aligned}$$

and

$$\begin{aligned} C &= \sqrt{\frac{2H}{(H-\frac{1}{2})\mathrm{Beta}(H-\frac{1}{2},2-2H)}} = \sqrt{\frac{(2H)\Gamma(\frac{3}{2}-H)}{\Gamma(2-2H)\Gamma(H+\frac{1}{2})}}, \\ c &= \frac{1}{\mathrm{Beta}(H+\frac{1}{2},\frac{3}{2}-H)} = \frac{\sin((H-\frac{1}{2})\pi)}{(H-\frac{1}{2})\pi}, \\ C' &= \frac{c}{C}. \end{aligned}$$

We use the previous formulas to present the processes $Y^{(\alpha)}$ and $Y^{(1)}$ in the integral forms with respect to Brownian motion. The key to these presentations is (3.1) in Definition 3.1.

Making the substitution $u = \tau_s = \frac{H e^{\frac{\alpha s}{H}}}{\alpha}$, we obtain

$$\begin{aligned}
Y_t^{(\alpha)} &= \int_0^t e^{-\alpha s} dZ_{\tau_s} \\
&= \left(\frac{H}{\alpha}\right)^H \int_{\frac{H}{\alpha}}^{\tau_t} u^{-H} dZ_u \\
&= \left(\frac{H}{\alpha}\right)^H \left(\tau_t^{-H} Z_{\tau_t} - \left(\frac{H}{\alpha}\right)^{-H} Z_{\frac{H}{\alpha}} - \int_{\frac{H}{\alpha}}^{\tau_t} H u^{-(H+1)} Z_u du \right).
\end{aligned}$$

Using the formula (3.13) for fractional Brownian motion, we infer that

$$Y_t^{(\alpha)} = \left(\frac{H}{\alpha}\right)^H C \left(I_1 - I_2 + I_3\right),$$

where

$$\begin{aligned}
I_1 &:= \tau_t^{-H} \int_0^{\tau_t} K(\tau_t, s) dB_s \\
I_2 &:= \left(\frac{H}{\alpha}\right)^{-H} \int_0^{\frac{H}{\alpha}} K\left(\frac{H}{\alpha}, s\right) dB_s \\
I_3 &:= \int_{\frac{H}{\alpha}}^{\tau_t} H u^{-(H+1)} \int_0^u K(u,s) dB_s du.
\end{aligned}$$

Next we split I_3 into two parts and change the order of the integration. This is allowed by the Fubini-type theorem for stochastic integrals if we find an integrable function $f(u)$ such that

$$|H u^{-(H+1)} K(u,s)| \leq f(u)$$

see, for example, Ikeda and Watanabe [26]. To achieve this, we make some approximations. Let $s \leq u$. Then we compute

$$\begin{aligned}
H u^{-(H+1)} K(u,s) &= H u^{-(H+1)} (H - \frac{1}{2}) s^{H-\frac{1}{2}} \int_s^u r^{H-\frac{1}{2}} (r-s)^{H-\frac{3}{2}} dr \\
&\leq H u^{-(H+1)} (H - \frac{1}{2}) u^{H-\frac{1}{2}} u^{H-\frac{1}{2}} \int_s^u (r-s)^{H-\frac{3}{2}} dr \\
&= H u^{-(H+1)} u^{2(H-\frac{1}{2})} (u-s)^{H-\frac{1}{2}} \\
&\leq H u^{-(H+1)} u^{2(H-\frac{1}{2})} u^{H-\frac{1}{2}} \\
&= H u^{2H-\frac{5}{2}} \\
&=: f(u)
\end{aligned}$$

and

$$\int_{\frac{H}{\alpha}}^{\tau_t} u^{2H-\frac{5}{2}} du = \frac{\tau_t^{2H-\frac{3}{2}} - \left(\frac{H}{\alpha} t\right)^{2H-\frac{3}{2}}}{2H-\frac{3}{2}} < \infty.$$

Hence we may change the order of integrations, yielding

$$\begin{aligned} I_3 &= \int_{\frac{H}{\alpha}}^{\tau_t} Hu^{-(H+1)} \int_0^{\frac{H}{\alpha}} K(u,s) dB_s du + \int_{\frac{H}{\alpha}}^{\tau_t} Hu^{-(H+1)} \int_{\frac{H}{\alpha}}^{u} K(u,s) dB_s du \\ &= \int_0^{\frac{H}{\alpha}} \int_{\frac{H}{\alpha}}^{\tau_t} Hu^{-(H+1)} K(u,s) du dB_s + \int_{\frac{H}{\alpha}}^{\tau_t} \int_s^{\tau_t} Hu^{-(H+1)} K(u,s) du dB_s. \end{aligned}$$

Then, using the integration by parts and denoting $K(t,s) := \int_s^t k(u,s)du$, where

$$k(t,s) := (H - \frac{1}{2}) \left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{3}{2}},$$

we obtain

$$\begin{aligned} I_3 &= \int_0^{\frac{H}{\alpha}} \left(\frac{H}{\alpha}\right)^{-H} K\left(\frac{H}{\alpha}, s\right) - \tau_t{}^{-H} K(\tau_t, s) dB_s \\ &\quad - \int_0^{\frac{H}{\alpha}} \int_{\frac{H}{\alpha}}^{\tau_t} \left(-u^{-H} \left(k(\tau_t, u) - k\left(\frac{H}{\alpha}, u\right)\right)\right) du dB_s \\ &\quad + \int_{\frac{H}{\alpha}}^{\tau_t} s^{-H} K(s,s) - \tau_t^{-H} K(\tau_t, s) dB_s \\ &\quad - \int_{\frac{H}{\alpha}}^{\tau_t} \int_s^{\tau_t} \left(u^{-H} \left(k(\tau_t, u) - k(s,u)\right)\right) du dB_s. \end{aligned}$$

Simplifying, we infer

$$\begin{aligned} Y_t^{(\alpha)} &= \left(\frac{H}{\alpha}\right)^H C \left(\int_0^{\frac{H}{\alpha}} \int_{\frac{H}{\alpha}}^{\tau_t} u^{-H} k(\tau_t, u) du dB_s \right. \\ &\quad - \int_0^{\frac{H}{\alpha}} \int_{\frac{H}{\alpha}}^{\tau_t} u^{-H} k\left(\frac{H}{\alpha}, u\right) du dB_s \\ &\quad \left. - \int_{\frac{H}{\alpha}}^{\tau_t} \int_s^{\tau_t} u^{-H} k\left(\tau_t, u\right) du dB_s + \int_{\frac{H}{\alpha}}^{\tau_t} \int_s^{\tau_t} u^{-H} k\left(s, u\right) du dB_s \right). \end{aligned}$$

Since there are integrals which are independent on s and $\int_a^b dB_s = B_b - B_a$, we conclude

$$Y_t^{(\alpha)} = \left(\frac{H}{\alpha}\right)^H C\left(B_{\frac{H}{\alpha}}\left(\int_{\frac{H}{\alpha}}^{\tau_t} u^{-H}\left(k(\tau_t,u) - k\left(\frac{H}{\alpha},u\right)du\right)\right)\right.$$
$$\left. + \int_{\frac{H}{\alpha}}^{\tau_t}\int_s^{\tau_t} u^{-H}\left(k\left(s,u\right) - k\left(\tau_t,u\right)\right) dudB_s\right).$$

If $\alpha = 1$, we infer directly $Y^{(1)}$ as follows

$$Y_t^{(1)} = H^H C\left(B_H\left(\int_H^{\tau_t^{(1)}} u^{-H}\left(k(\tau_t^{(1)},u) - k\left(H,u\right)du\right)\right)\right.$$
$$\left. + \int_H^{\tau_t^{(1)}}\int_s^{\tau_t^{(1)}} u^{-H}\left(k\left(s,u\right) - k\left(\tau_t^{(1)},u\right)\right) dudB_s\right).$$

Collecting the previous results we have proved the following proposition

Proposition 3.9. *Let $H > \frac{1}{2}$ and*

$$\begin{aligned} k^{(\alpha)}(t,s,u) &:= k(t,u) - k(s,u) \\ &= (H - \frac{1}{2})u^{\frac{1}{2}-H}\left(t^{H-\frac{1}{2}}(t-u)^{H-\frac{3}{2}} - s^{H-\frac{1}{2}}(s-u)^{H-\frac{3}{2}}\right), \end{aligned}$$

then the Volterra representations of $Y_t^{(\alpha)}$ and $Y_t^{(1)}$ are

$$Y_t^{(\alpha)} = \left(\frac{H}{\alpha}\right)^H C\Bigg(B_{\frac{H}{\alpha}}\Big(\int_{\frac{H}{\alpha}}^{\tau_t} u^{-H}k^{(\alpha)}(\tau_t, \frac{H}{\alpha}, u)du\Big)$$
$$+ \int_{\frac{H}{\alpha}}^{\tau_t}\int_s^{\tau_t} u^{-H}k^{(\alpha)}(s,\tau_t,u)dudB_s\Bigg)$$

$$Y_t^{(1)} = H^H C\Bigg(B_H\Big(\int_H^{\tau_t^{(1)}} u^{-H}k^{(\alpha)}(\tau_t^{(1)}, H, u)du\Big)$$
$$+ \int_H^{\tau_t^{(1)}}\int_s^{\tau_t^{(1)}} u^{-H}k^{(\alpha)}(s,\tau_t^{(1)},u)dudB_s\Bigg).$$

3.1.5 Hölder continuity of the order $\beta < H$

Knowing that both processes $Y^{(\alpha)}$ and $U^{(D,\gamma)}$ have the same kind of continuity properties is useful. We prove that they are both locally Hölder continuous of any order $\beta < H$. This result is strong enough for us but in Zähle [58] there is a more general result. We recall that a definition of locally Hölder continuity is given in Definition 1.15. Azmoodeh and Viitasaari used the same kind of approximations of the fractional Ornstein-Uhlenbeck process of the second kind in [3, Lemma 2.4] as we use in the proof of Proposition 3.11.

Proposition 3.10 ([29], Prop. 3.4). *The sample paths of the process $Y^{(\alpha)}$ are locally Hölder continuous of any order $\beta < H$.*

Proof. Applying (3.4), we obtain

$$\begin{aligned} Y_t^{(\alpha)} &= \mathrm{e}^{-\alpha t} Z_{\tau_t} - Z_{\frac{H}{\alpha}} + \alpha \int_0^t \mathrm{e}^{-\alpha s} Z_{\tau_s} ds \\ &= X_t^{(D,\alpha)} - X_0^{(D,\alpha)} + \alpha \int_0^t X_s^{(D,\alpha)} ds, \end{aligned} \tag{3.15}$$

where $X_t^{(D,\alpha)} = \mathrm{e}^{-\alpha t} Z_{\tau_t}$ is the Doob transformation of fBm and $\tau_t = \frac{\mathrm{e}^{\frac{\alpha t}{H}} H}{\alpha}$. We first prove that $X^{(D,\alpha)}$ is locally Hölder continuous of any order $\beta < H$. We just compute

$$\begin{aligned} \frac{\left|X_t^{(D,\alpha)} - X_s^{(D,\alpha)}\right|}{|t-s|^\beta} &= \frac{|\mathrm{e}^{-\alpha t} Z_{\tau_t} - \mathrm{e}^{-\alpha s} Z_{\tau_s}|}{|t-s|^\beta} \\ &= \frac{|\mathrm{e}^{-\alpha t} Z_{\tau_t} - \mathrm{e}^{-\alpha s} Z_{\tau_t} + \mathrm{e}^{-\alpha s}\left(Z_{\tau_t} - Z_{\tau_s}\right)|}{|t-s|^\beta} \\ &\leq \frac{|\mathrm{e}^{-\alpha t} - \mathrm{e}^{-\alpha s}|\,|Z_{\tau_t}|}{|t-s|^\beta} + \frac{\mathrm{e}^{-\alpha s}\,|Z_{\tau_t} - Z_{\tau_s}|}{|t-s|^\beta}, \end{aligned} \tag{3.16}$$

where $\beta > 0$ and $0 < s, t < T$. The first part of the sum (3.16) must be

$$\frac{|\mathrm{e}^{-\alpha t} - \mathrm{e}^{-\alpha s}|\,|Z_{\tau_t}|}{|t-s|^\beta} \leq C_T,$$

when $\beta < H$. By the Mean Value Theorem there exists $\xi \in (s,t)$ such that

$$|\mathrm{e}^{-\alpha t} - \mathrm{e}^{-\alpha s}| = |-\alpha \mathrm{e}^{-\alpha \xi}(t-s)| \leq \alpha|t-s|.$$

So

$$\frac{|\mathrm{e}^{-\alpha t} - \mathrm{e}^{-\alpha s}|}{|t-s|^\beta} \leq \alpha \frac{|t-s|}{|t-s|^\beta} = \alpha |t-s|^{1-\beta}.$$

Let $S_T = \sup_{0 \leq u \leq T} |Z_{\tau_u}|$, then we infer, since $\beta < H < 1$,

$$\begin{aligned} \frac{|\mathrm{e}^{-\alpha t} - \mathrm{e}^{-\alpha s}|\,|Z_{\tau_t}|}{|t-s|^\beta} &\leq \alpha S_T |t-s|^{1-\beta} \\ &\leq \alpha S_T T^{1-\beta} =: C_T. \end{aligned}$$

Next, we consider the second part of the sum (3.16)

$$\frac{\mathrm{e}^{-\alpha s}\left|Z_{\tau_t}-Z_{\tau_s}\right|}{|t-s|^{\beta}} = \frac{\left|Z_{\tau_t}-Z_{\tau_s}\right|}{\mathrm{e}^{\alpha s}\,|t-s|^{\beta}}.$$

Using the Mean Value Theorem we know that there exists $\xi \in (s,t)$ such that

$$\mathrm{e}^{\frac{\alpha t}{H}}-\mathrm{e}^{\frac{\alpha s}{H}}=\frac{\alpha}{H}\mathrm{e}^{\frac{\alpha \xi}{H}}(t-s).$$

Let ξ be the previous value, then it is possible to write

$$\frac{\mathrm{e}^{-\alpha s}\left|Z_{\tau_t}-Z_{\tau_s}\right|}{|t-s|^{\beta}} = \frac{\left|Z_{\tau_t}-Z_{\tau_s}\right|}{\mathrm{e}^{\alpha(s-\xi)}\left|\mathrm{e}^{\frac{\alpha\xi}{\beta}}(t-s)\right|^{\beta}}$$

and approximating it upwards, since $\beta < H$, we obtain

$$\frac{\mathrm{e}^{-\alpha s}\left|Z_{\tau_t}-Z_{\tau_s}\right|}{|t-s|^{\beta}} \leq \frac{\left|Z_{\tau_t}-Z_{\tau_s}\right|}{\mathrm{e}^{\alpha(s-\xi)}\left|\mathrm{e}^{\frac{\alpha\xi}{H}}(t-s)\right|^{\beta}}.$$

Now in (3.17) we use the already fixed $\xi \in (s,t)$, which we found using the Mean Value Theorem, such that

$$\frac{\left|Z_{\tau_t}-Z_{\tau_s}\right|}{\mathrm{e}^{\alpha(s-\xi)}\left|\mathrm{e}^{\frac{\alpha\xi}{H}}(t-s)\right|^{\beta}} = \frac{\left|Z_{\tau_t}-Z_{\tau_s}\right|}{\mathrm{e}^{\alpha(s-\xi)}\left|\frac{H}{\alpha}\left(\mathrm{e}^{\frac{\alpha t}{H}}-\mathrm{e}^{\frac{\alpha s}{H}}\right)\right|^{\beta}}$$

and we approximate further

$$\begin{aligned}\frac{\mathrm{e}^{-\alpha s}\left|Z_{\tau_t}-Z_{\tau_s}\right|}{|t-s|^{\beta}} &\leq \frac{\left|Z_{\tau_t}-Z_{\tau_s}\right|}{\mathrm{e}^{\alpha(s-\xi)}\left|\frac{H}{\alpha}\left(\mathrm{e}^{\frac{\alpha t}{H}}-\mathrm{e}^{\frac{\alpha s}{H}}\right)\right|^{\beta}} \\ &\leq \mathrm{e}^{\alpha T}\frac{\left|Z_{\tau_t}-Z_{\tau_s}\right|}{\left|\frac{H}{\alpha}\left(\mathrm{e}^{\frac{\alpha t}{H}}-\mathrm{e}^{\frac{\alpha s}{H}}\right)\right|^{\beta}} = K_T\frac{\left|Z_{\tau_t}-Z_{\tau_s}\right|}{\left|\tau_t-\tau_s\right|^{\beta}}.\end{aligned} \tag{3.17}$$

Hence we obtain

$$\frac{\left|X_t^{(D,\alpha)}-X_s^{(D,\alpha)}\right|}{|t-s|^{\beta}} \leq K_T\frac{\left|Z_{\tau_t}-Z_{\tau_s}\right|}{\left|\frac{\mathrm{e}^{\frac{\alpha t}{H}}H}{\alpha}-\frac{\mathrm{e}^{\frac{\alpha s}{H}}H}{\alpha}\right|^{\beta}}+C_T,$$

where K_T and C_T are random constants that are independent of s and t. Since fBm is locally Hölder continuous of the order $\beta < H$, we conclude that $X^{(D,\alpha)}$ is Hölder continuous of the order $\beta < H$. Last we consider integral term in (3.15). Since

$$\begin{aligned}\left|\int_s^t X_u^{(D,\alpha)}du\right| &\leq \sup_{s\leq u\leq t}|X_u^{(D,\alpha)}||t-s| \\ &\leq \sup_{1\leq u\leq T}|X_u^{(D,\alpha)}||t-s| =: C_T|t-s|,\end{aligned}$$

then $\int_s^t X_u^{(D,\alpha)} du$ is Hölder continuous of the order 1. And therefore it is also Hölder continuous of the order $\beta < 1$, since

$$\begin{aligned}
\frac{\left|\int_s^t X_u^{(D,\alpha)} du\right|}{|t-s|^{\beta}} &= \frac{|t-s|^{-\beta+1}\left|\int_s^t X_u^{(D,\alpha)} du\right|}{|t-s|} \\
&\leq |t-s|^{-\beta+1} C_T \\
&\leq T^{1-\beta} C_T.
\end{aligned}$$

Hence the sample paths of the process Y^{α} are locally Hölder continuous. □

Proposition 3.11 ([29], Prop. 3.4.). *The sample paths of the process $U^{(D,\gamma)}$ are locally Hölder continuous of any order $\beta < H$.*

Proof. Using exactly the same idea as in the proof of Proposition 3.10, we can prove that $U^{(D,\gamma)}$ is locally Hölder continuous of the order $\beta < H$, since we have the following connection between $U^{(D,\gamma)}$ and $Y^{(1)}$

$$U_t^{(D,\gamma)} - \mathrm{e}^{-\gamma t} U_0^{(D,\gamma)} = Y_t^{(1)} - \gamma \mathrm{e}^{-\gamma t} \int_0^t \mathrm{e}^{\gamma s} Y_s^{(1)} ds, \ t > 0.$$

Using the previous equation we start to calculate the difference of $U^{(D,\gamma)}$

$$\begin{aligned}
\left|U_t^{(D,\gamma)} - U_s^{(D,\gamma)}\right| &= \Bigg|(\mathrm{e}^{-\gamma t} - \mathrm{e}^{-\gamma s}) U_0^{(D,\gamma)} + Y_t^{(1)} - Y_s^{(1)} \\
&\quad - \left(\gamma \mathrm{e}^{-\gamma t} \int_0^t \mathrm{e}^{\gamma s} Y_s^{(1)} ds - \gamma \mathrm{e}^{-\gamma s} \int_0^s \mathrm{e}^{\gamma r} Y_r^{(1)} dr\right)\Bigg| \\
&\leq \left|\mathrm{e}^{-\gamma t} - \mathrm{e}^{-\gamma s}\right| \left|U_0^{(D,\gamma)}\right| + \left|Y_t^{(1)} - Y_s^{(1)}\right| \\
&\quad + \left|\gamma\left(\int_0^t \mathrm{e}^{-\gamma p} Y_{t-p}^{(1)} dp - \int_0^s \mathrm{e}^{-\gamma p} Y_{s-p}^{(1)} dp\right)\right| \\
&\leq \left|\mathrm{e}^{-\gamma t} - \mathrm{e}^{-\gamma s}\right| \left|U_0^{(D,\gamma)}\right| + \left|Y_t^{(1)} - Y_s^{(1)}\right| \\
&\quad + \left|\gamma\left(\int_0^s \mathrm{e}^{-\gamma p}\left(Y_{t-p}^{(1)} - Y_{s-p}^{(1)}\right) dp + \int_s^t \mathrm{e}^{-\gamma p} Y_{t-p}^{(1)} dp\right)\right|,
\end{aligned}$$

where

- using the Mean Value Theorem we know that there is $\xi \in (s,t)$ such that

  $$\left|\mathrm{e}^{-\gamma t} - \mathrm{e}^{-\gamma s}\right| = \left|-\gamma \mathrm{e}^{-\gamma \xi} |t-s|\right|.$$

  Denoting $C_{T_0} := \gamma \left|U_0^{(D,\gamma)}\right|$ we infer

  $$\begin{aligned}
  \left|\mathrm{e}^{-\gamma t} - \mathrm{e}^{-\gamma s}\right| \left|U_0^{(D,\gamma)}\right| &\leq \left|-\mathrm{e}^{-\gamma\xi} C_{T_0} |t-s|\right| \\
  &\leq C_{T_0} |t-s|,
  \end{aligned}$$

- using the Hölder continuity of $Y^{(1)}$ we obtain

$$\left|Y_t^{(1)} - Y_s^{(1)}\right| \leq C_{T_1} |t-s|^{\beta},$$

- using again the Hölder continuity of $Y^{(1)}$ we deduce

$$\begin{aligned}\left|\gamma \int_0^s \mathrm{e}^{-\gamma p} \left(Y_{t-p}^{(1)} - Y_{s-p}^{(1)}\right) dp\right| &\leq \left|\gamma \int_0^s \mathrm{e}^{-\gamma p} C_{T_2} |t-s|^{\beta} dp\right| \\ &\leq C_{T_2} |t-s|^{\beta} (1 - \mathrm{e}^{-\gamma s}) \\ &\leq C_{T_2} |t-s|^{\beta},\end{aligned}$$

- lastly the continuity of $Y^{(1)}$ at t, implies that

$$\left|\gamma \int_s^t \mathrm{e}^{-\gamma p} Y_{t-p}^{(1)} dp\right| \leq \gamma \sup_{s \leq p \leq t} \left|Y_{t-p}^{(1)}\right| \int_s^t \mathrm{e}^{-\gamma p} dp$$

and

$$\begin{aligned}\left|\gamma \int_s^t \mathrm{e}^{-\gamma p} Y_{t-p}^{(1)} dp\right| &\leq \sup_{0 \leq p \leq T} \left|Y_{t-p}^{(1)}\right| \mathrm{e}^{-\gamma \xi} |t-s| \\ &\leq C_{T_3} |t-s|.\end{aligned}$$

Combining the previous approximations, we infer

$$\begin{aligned}\left|U_t^{(D,\gamma)} - U_s^{(D,\gamma)}\right| &\leq \left|\mathrm{e}^{-\gamma t} - \mathrm{e}^{-\gamma s}\right| \left|U_0^{(D,\gamma)}\right| + \left|Y_t^{(1)} - Y_s^{(1)}\right| \\ &\quad + \left|\gamma \left(\int_0^s \mathrm{e}^{-\gamma p} \left(Y_{t-p}^{(1)} - Y_{s-p}^{(1)}\right) dp + \int_s^t \mathrm{e}^{-\gamma p} Y_{t-p}^{(1)} dp\right)\right| \\ &\leq C_{T_0} |t-s| + C_{T_1} |t-s|^{\beta} + C_{T_2} |t-s|^{\beta} + C_{T_3} |t-s|.\end{aligned}$$

Denoting $C_{T_0} + C_{T_3} =: C_{T_a}$ and $C_{T_1} + C_{T_2} =: C_{T_b}$, we are able to write

$$\begin{aligned}\frac{\left|U_t^{(D,\gamma)} - U_s^{(D,\gamma)}\right|}{|t-s|^{\beta}} &\leq C_{T_a} |t-s|^{1-\beta} + C_{T_b} \\ &\leq C_{T_a} T^{1-\beta} + C_{T_b}\end{aligned}$$

As we have seen above, $U^{(D,\gamma)}$ is Hölder continuous of the order β if $Y^{(1)}$ is Hölder continuous of the order β.

□

3.2 Covariance kernels

In this section we consider the kernel representation of fOU processes and the driving processes of fOU processes. Since fractional Brownian motion is behind all these processes the kernel representation of fractional Brownian motion we recall here

$$\mathbf{E}\left((Z_{t_2}-Z_{t_1})(Z_{s_2}-Z_{s_1})\right) = \int_{s_1}^{s_2}\int_{t_1}^{t_2}(2H-1)H(u-v)^{2H-2}dudv,$$

for all $H \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$.

3.2.1 The Doob transformation of fBm (fOU)

Our aim is to find the kernel representation of the covariance of the Doob transformation. Differentiating twice the covariance of the Doob transformation of (2.3) given in the proof of Proposition 2.1 yields

$$\begin{aligned}
&\frac{\partial}{\partial t}\frac{\partial}{\partial s}\left(\mathbf{E}\left(X_t^{(D,\alpha)}X_s^{(D,\alpha)}\right)\right)\\
&\quad= \frac{\partial}{\partial t}\frac{\partial}{\partial s}\left(\mathrm{e}^{-\alpha(t+s)}\frac{1}{2}\left(\frac{H}{\alpha}\right)^{2H}\left(\mathrm{e}^{2\alpha t}+\mathrm{e}^{2\alpha s}-\mathrm{e}^{2\alpha t}\left(1-\mathrm{e}^{-\frac{\alpha(t-s)}{H}}\right)^{2H}\right)\right)\\
&\quad= \frac{1}{2}\left(\frac{H}{\alpha}\right)^{2H}\mathrm{e}^{-\alpha(t+s)}\alpha^2\left(-\mathrm{e}^{2\alpha s}-\mathrm{e}^{2\alpha t}+\mathrm{e}^{2\alpha t}\left(1-\mathrm{e}^{\frac{-\alpha(t-s)}{H}}\right)^{2H}\right.\\
&\qquad\qquad+\left(4-\frac{2}{H}\right)\mathrm{e}^{2\alpha t}\left(1-\mathrm{e}^{\frac{-\alpha(t-s)}{H}}\right)^{2H-1}\mathrm{e}^{\frac{\alpha(t-s)}{H}}\\
&\qquad\qquad\left.+\left(4-\frac{2}{H}\right)\mathrm{e}^{2\alpha t}\left(1-\mathrm{e}^{\frac{-\alpha(t-s)}{H}}\right)^{2H-2}\left(\mathrm{e}^{\frac{\alpha(t-s)}{H}}\right)^2\right).
\end{aligned}$$

This leads to

$$\begin{aligned}
&\frac{\partial}{\partial t}\frac{\partial}{\partial s}\left(\mathbf{E}\left(X_t^{(D,\alpha)}X_s^{(D,\alpha)}\right)\right)\\
&\quad= \frac{\alpha^2}{2}\left(\frac{H}{\alpha}\right)^{2H}\left(-\mathrm{e}^{-\alpha(t-s)}-\mathrm{e}^{\alpha(t-s)}\right)\\
&\qquad+\frac{\alpha^2}{2}\left(\frac{H}{\alpha}\right)^{2H}\mathrm{e}^{\alpha(t-s)}\left(1-\mathrm{e}^{\frac{\alpha(t-s)}{H}}\right)^{2H}\times\\
&\qquad\left(1+\left(\frac{1}{H}-2\right)\left(\frac{\mathrm{e}^{\frac{\alpha(t-s)}{H}}}{1-\mathrm{e}^{\frac{\alpha(t-s)}{H}}}-\left(\frac{\mathrm{e}^{\frac{\alpha(t-s)}{H}}}{1-\mathrm{e}^{\frac{\alpha(t-s)}{H}}}\right)^2\right)\right).
\end{aligned}$$

The kernel is quite complicated. Nevertheless, it is still good to know the kernel representation of the Doob transformation.

We have just proved the next theorem

Proposition 3.12. *For $H \in (0,1)$ and $\alpha > 0$, the covariance kernel of the increments* *of* $X^{(D,\alpha)}$ *is*

$$\begin{aligned} k_{X^{(D,\alpha)}}(u,v) &= \frac{\alpha^2}{2}\left(\frac{H}{\alpha}\right)^{2H}\left(-\mathrm{e}^{-\alpha(v-u)}-\mathrm{e}^{\alpha(v-u)}\right) \\ &\quad+\frac{\alpha^2}{2}\left(\frac{H}{\alpha}\right)^{2H}\mathrm{e}^{\alpha(v-u)}\left(1-\mathrm{e}^{\frac{\alpha(v-u)}{H}}\right)^{2H}\times \\ &\qquad\left(1+\left(\frac{1}{H}-2\right)\left(\frac{\mathrm{e}^{\frac{\alpha(v-u)}{H}}}{1-\mathrm{e}^{\frac{\alpha(v-u)}{H}}}-\left(\frac{\mathrm{e}^{\frac{\alpha(v-u)}{H}}}{1-\mathrm{e}^{\frac{\alpha(v-u)}{H}}}\right)^2\right)\right), \end{aligned}$$

and the covariance of the increments of $X^{(D,\alpha)}$ *is*

$$\mathbf{E}\left(\left(X_{t_2}^{(D,\alpha)}-X_{t_1}^{(D,\alpha)}\right)\left(X_{s_2}^{(D,\alpha)}-X_{s_1}^{(D,\alpha)}\right)\right)=\int_{t_1}^{t_2}\int_{s_1}^{s_2}k_{X^{(D,\alpha)}}(u,v)dvdu, \tag{3.18}$$

where $s_1 < s_2$, $t_1 < t_2$.

3.2.2 Covariance kernels of $Y^{(\alpha)}$ and $Y^{(1)}$, when $H > \frac{1}{2}$

We note that in the case $H \in (\frac{1}{2}, 1)$, the increments of fBm are positively correlated and the covariance kernel has some nice integrability properties, see, for example, Pipiras and Taqqu [51, p.16] Equation (4.1) and discussion after that.

Proposition 3.13 ([29], Prop. 3.5.)**.** *For* $H \in (\frac{1}{2}, 1)$, *the covariance kernel of the increments of* $Y^{(\alpha)}$ *is*

$$\begin{aligned} &\mathbf{E}\left(\left(Y_{t_2}^{(\alpha)}-Y_{t_1}^{(\alpha)}\right)\left(Y_{s_2}^{(\alpha)}-Y_{s_1}^{(\alpha)}\right)\right) \\ &= C(\alpha,H)\int_{t_1}^{t_2}\int_{s_1}^{s_2}\frac{\mathrm{e}^{-\frac{\alpha(1-H)(u-v)}{H}}}{\left|1-\mathrm{e}^{-\frac{\alpha(u-v)}{H}}\right|^{2-2H}}dudv, \end{aligned} \tag{3.19}$$

where $s_1 < s_2$, $t_1 < t_2$ *and*

$$C(\alpha,H) := H(2H-1)\left(\frac{\alpha}{H}\right)^{2-2H}.$$

Proof. We need Proposition 2.2 of Gripenberg and Norros [22], stating that

$$\begin{aligned} &\mathbf{E}\left(\int_{\mathbb{R}}f(s)dZ_s\int_{\mathbb{R}}g(t)dZ_t\right) \\ &= H(2H-1)\int_{\mathbb{R}}\int_{\mathbb{R}}f(s)g(t)|s-t|^{2H-2}dtds, \end{aligned} \tag{3.20}$$

when $H \in (\frac{1}{2}, 1)$ and $f, g \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$.

Note that the integrals in (3.20) are over the real axis, but we also have a similar result for bounded intervals when using the indicator function

$$\int_a^b f(t)dt = \int_{\mathbb{R}} f(t)\mathbf{1}_{[a,b]}dt.$$

We may indeed modify the process Y^α to a form that fits into (3.20). Substituting $u := \dfrac{e^{\frac{\alpha s}{H}} H}{\alpha}$, we obtain

$$e^{-\alpha s} = \left(\frac{1}{e^{\frac{\alpha s}{H}}}\right)^H = \left(\frac{\alpha}{H}\right)^{-H} u^{-H}.$$

Therefore

$$\begin{aligned} Y_t^\alpha &= \int_0^t e^{-\alpha s} dZ_{\tau_s} \\ &= \left(\frac{H}{\alpha}\right)^H \int_{\frac{H}{\alpha}}^{\frac{e^{\frac{\alpha t}{H}} H}{\alpha}} u^{-H} dZ_u. \end{aligned}$$

We consider the function

$$f(s) = \begin{cases} s^{-H}, & \text{if } \tau_0 < s < \tau_t \\ 0, & \text{otherwise,} \end{cases}$$

where $\tau_t = \dfrac{e^{\frac{\alpha t}{H}} H}{\alpha}$. Since in the case $H \in (\frac{1}{2}, 1)$ the function u^{-H} belongs to $L^2(I) \cap L^1(I)$, where $I = [\tau_0, \tau_t] \subset \mathbb{R}$ we can change the variables two times, to obtain

$$\begin{aligned} &\mathbf{E}\left(\left(Y_{t_2}^{(\alpha)} - Y_{t_1}^{(\alpha)}\right)\left(Y_{s_2}^{(\alpha)} - Y_{s_1}^{(\alpha)}\right)\right) \\ &= \mathbf{E}\left(\left(\left(\frac{H}{\alpha}\right)^H \int_{\tau_{t_1}}^{\tau_{t_2}} p^{-H} dZ_p\right)\left(\left(\frac{H}{\alpha}\right)^H \int_{\tau_{s_1}}^{\tau_{s_2}} r^{-H} dZ_r\right)\right) \\ &= \left(\frac{H}{\alpha}\right)^{2H} H(2H-1) \int_{\tau_{t_1}}^{\tau_{t_2}} \int_{\tau_{s_1}}^{\tau_{s_2}} p^{-H} r^{-H} |p-r|^{2H-2} dp dr \\ &= \left(\frac{\alpha}{H}\right)^{2-2H} H(2H-1) \int_{t_1}^{t_2} \int_{s_1}^{s_2} e^{-\frac{1-H}{H}\alpha(t-s)} \left|1 - e^{-\frac{\alpha(t-s)}{H}}\right|^{2H-2} dt ds, \end{aligned}$$

thereby completing the proof. □

The next corollary is quite obvious, but the proof is a nice example of the behaviour of the exponential function.

Corollary 3.14 ([29], Prop. 3.5.). *In the case $H \in (\frac{1}{2}, 1)$, the kernel in the representation of the covariance of the increments of the process $Y^{(\alpha)}$ is symmetric.*

Proof. We recall the kernel in the representation of the covariance of $Y^{(\alpha)}$, given in Proposition 3.13,

$$r_{\alpha,H}(u,v) := C(\alpha, H) \frac{e^{-\frac{\alpha(1-H)(u-v)}{H}}}{\left|1 - e^{-\frac{\alpha(u-v)}{H}}\right|^{2-2H}}.$$

Note that $r_{\alpha,H}(u,v)$ is symmetric. This property follows from the computations

$$\begin{aligned}
r_{\alpha,H}(u,v) &= C(\alpha,H)\frac{e^{-\frac{\alpha(1-H)(u-v)}{H}}}{\left|1-e^{-\frac{\alpha(u-v)}{H}}\right|^{2-2H}} = C(\alpha,H)\frac{e^{\frac{\alpha(1-H)(v-u)}{H}}}{\left|1-e^{\frac{\alpha(v-u)}{H}}\right|^{2-2H}} \\
&= C(\alpha,H)\frac{e^{\frac{\alpha(1-H)(v-u)}{H}}}{\left|1-\frac{1}{e^{-\frac{\alpha(v-u)}{H}}}\right|^{2-2H}} = C(\alpha,H)\frac{e^{\frac{\alpha(1-H)(v-u)}{H}}}{\left|\frac{e^{-\frac{\alpha(v-u)}{H}}-1}{e^{-\frac{\alpha(v-u)}{H}}}\right|^{2-2H}} \\
&= C(\alpha,H)\frac{e^{\frac{\alpha(1-H)(v-u)}{H}-\frac{2\alpha(1-H)(v-u)}{H}}}{\left|1-e^{-\frac{\alpha(v-u)}{H}}\right|^{2-2H}} \\
&= C(\alpha,H)\frac{e^{-\frac{\alpha(1-H)(v-u)}{H}}}{\left|1-e^{-\frac{\alpha(v-u)}{H}}\right|^{2-2H}} = r_{\alpha,H}(v,u),
\end{aligned}$$

which establish the desired property. □

3.2.3 Covariance kernel of fOU(2), when $H > \frac{1}{2}$

We assume again that $H \in (\frac{1}{2}, 1)$ since the covariance kernel has some integrability properties, see, for example, Pipiras and Taqqu [51, p.16]. are valid in this interval.

Proposition 3.15 ([29], Prop. 3.10.). *The covariance of the process $U^{(D,\gamma)}$, for $H \in (\frac{1}{2}, 1)$, (fOU(2)) has the kernel representation*

$$\begin{aligned}
&\mathbf{E}\left(U_t^{(D,\gamma)}U_s^{(D,\gamma)}\right) \qquad (3.21)\\
&= (2H-1)H^{2H-1}e^{-\gamma(t+s)}\int_{-\infty}^{t}\int_{-\infty}^{s}\frac{e^{(\gamma-1+\frac{1}{H})(u+v)}}{\left|e^{\frac{u}{H}}-e^{\frac{v}{H}}\right|^{2(1-H)}}dudv.
\end{aligned}$$

Proof. The proof has the same elements as the proof of Proposition 3.13. However, the situation is not identical because, now, we consider the expectation of the processes instead of the expectation of their differences. Therefore, we do not have integrals over any restricted interval, instead we have double integrals over unrestricted intervals. To state that the property (3.20) holds for the functions f and g we need an extended version of this property, stated in [51, Eq. (4.1)]. For using this Equation (4.1) of Pipiras and Taqqu the functions f and g have to satisfy the condition

$$\int_{\mathbb{R}}\int_{\mathbb{R}}|f(s)||g(t)||s-t|^{2H-2}dtds < \infty. \qquad (3.22)$$

We recall that in Definition 3.6 of fOU(2), the process $U^{(D,\gamma)}$ is given by

$$U_t^{(D,\gamma)} \;=\; e^{-\gamma t}\int_{-\infty}^{t} e^{(\gamma-1)s}dZ_{\tau_s^{(1)}},$$

where $\tau_t^{(1)} = H\mathrm{e}^{\frac{t}{H}}$. Changing the variable u by $H\mathrm{e}^{\frac{s}{H}}$, we infer

$$\begin{aligned} U_t^{(D,\gamma)} &= \mathrm{e}^{-\gamma t} \int_0^{\tau_t^{(1)}} \left(\left(\frac{u}{H} \right)^H \right)^{\gamma-1} dZ_u \\ &= \mathrm{e}^{-\gamma t} H^{-H(\gamma-1)} \int_0^{\tau_t^{(1)}} u^{H(\gamma-1)} dZ_u. \end{aligned}$$

Substitution is allowed, since the integral is the pathwise Riemann-Stieltjes integral. We still have to check that the condition (3.22) is valid for the functions $f(u) = g(u) = u^{H(\gamma-1)} \mathbf{1}_{(0,\tau_t^{(1)})}(u)$. We make the following calculations

$$\begin{aligned} &\int_{\mathbb{R}} \int_{\mathbb{R}} |f(p)||g(r)||p-r|^{2H-2} dp dr \\ &= \int_0^{\tau_t^{(1)}} \int_0^{\tau_t^{(1)}} p^{H(\gamma-1)} r^{H(\gamma-1)} |p-r|^{2H-2} dp dr \\ &= \int_0^1 \int_0^1 (\tau_t^{(1)} u)^{H(\gamma-1)} (\tau_t^{(1)} v)^{H(\gamma-1)} |\tau_t^{(1)} u - \tau_t^{(1)} v|^{2H-2} (\tau_t^{(1)})^2 du dv \\ &= \left(\tau_t^{(1)} \right)^{2H\gamma} \int_0^1 \int_0^1 (uv)^{H(\gamma-1)} |u-v|^{2H-2} du dv, \end{aligned}$$

and by the symmetry of the integrand we infer

$$\begin{aligned} &\int_{\mathbb{R}} \int_{\mathbb{R}} |f(p)||g(r)||p-r|^{2H-2} dp dr \\ &= 2 \left(\tau_t^{(1)} \right)^{2H\gamma} \int_0^1 du\, u^{H(\gamma-1)} \int_0^u v^{H(\gamma-1)} (u-v)^{2H-2} dv \\ &= 2 \left(\tau_t^{(1)} \right)^{2H\gamma} \int_0^1 du\, u^{H(\gamma-1)} \int_0^1 (uw)^{H(\gamma-1)} (u-uw)^{2H-2} u dw \\ &= 2 \left(\tau_t^{(1)} \right)^{2H\gamma} \int_0^1 du\, u^{H(\gamma-1)} u^{H(\gamma+1)-1} \int_0^1 w^{H(\gamma-1)} (1-w)^{2H-2} dw \\ &= 2 \left(\tau_t^{(1)} \right)^{2H\gamma} \int_0^1 du\, u^{H(\gamma-1)} u^{H(\gamma+1)-1} \mathrm{Beta}\left(H(\gamma-1)+1, 2H-1 \right) \\ &= \frac{\left(\tau_t^{(1)} \right)^{2H\gamma}}{\gamma H} \mathrm{Beta}\left(H(\gamma-1)+1, 2H-1 \right) < \infty, \end{aligned}$$

where Beta stands for the Beta function $\mathrm{Beta}(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1} dx$, which is defined and finite for any positive a and b.

Hence, we can calculate the covariance as follows

$$
\begin{aligned}
&\mathbf{E}\left(U_t^{(D,\gamma)}U_s^{(D,\gamma)}\right)\\
&= \mathbf{E}\left(\left(\mathrm{e}^{-\gamma t}H^{-H(\gamma-1)}\int_0^{\tau_t^{(1)}} u^{H(\gamma-1)}dZ_u\right)\left(\mathrm{e}^{-\gamma s}H^{-H(\gamma-1)}\int_0^{\tau_s^{(1)}} v^{H(\gamma-1)}dZ_v\right)\right)\\
&= \mathrm{e}^{-\gamma(t+s)}H^{-2H(\gamma-1)}H(2H-1)\int_0^{\tau_t^{(1)}}\int_0^{\tau_s^{(1)}} u^{H(\gamma-1)}v^{H(\gamma-1)}|u-v|^{2H-2}dudv.
\end{aligned}
$$

Substituting u with $H\mathrm{e}^{\frac{p}{H}}$ and v by $H\mathrm{e}^{\frac{r}{H}}$, we obtain

$$
\begin{aligned}
&\mathbf{E}\left(U_t^{(D,\gamma)}U_s^{(D,\gamma)}\right)\\
&= \mathrm{e}^{-\gamma(t+s)}H^{-2H(\gamma-1)}H(2H-1)\times\\
&\int_0^t\int_0^s \left(H\mathrm{e}^{\frac{p}{H}}\right)^{H(\gamma-1)}\left(H\mathrm{e}^{\frac{r}{H}}\right)^{H(\gamma-1)}|H\mathrm{e}^{\frac{p}{H}}-H\mathrm{e}^{\frac{r}{H}}|^{2H-2}\mathrm{e}^{\frac{p}{H}}\mathrm{e}^{\frac{r}{H}}dpdr\\
&= H(2H-1)H^{(2H-2)}\mathrm{e}^{-\gamma(t+s)}\int_0^t\int_0^s \mathrm{e}^{(\gamma-1+\frac{1}{H})(p+r)}|\mathrm{e}^{\frac{p}{H}}-\mathrm{e}^{\frac{r}{H}}|^{2H-2}dpdr,
\end{aligned}
$$

verifying the assertion. □

3.2.4 Covariance and variance of $Y^{(\alpha)}$

We recall the definition of $Y^{(\alpha)}$ of (3.1)

$$
Y_t^{(\alpha)} = \int_0^t \mathrm{e}^{-\alpha s}dZ_{\tau_s},
$$

where $\tau_s = \frac{H\mathrm{e}^{\frac{\alpha s}{H}}}{\alpha}$. In this section we consider covariance of $Y^{(\alpha)}$ and variance of $Y^{(\alpha)}$ and increments of $Y^{(\alpha)}$. We also examine the asymptotic bahaviours of variance and covariance, when $t \to \infty$. For the sake of readability, we include many propositions with their brief proofs.

Corollary 3.16 ([29], Cor. 3.7.)**.** *In the case $H \in (\frac{1}{2}, 1)$, the increments of $Y^{(\alpha)}$ are positively correlated.*

Proof. The coefficient in Proposition 3.13

$$
C(\alpha, H) = H(2H-1)\left(\frac{\alpha}{H}\right)^{2H-2},
$$

is always positive since $H \in (\frac{1}{2}, 1)$. The kernel in Proposition 3.13

$$
\frac{\mathrm{e}^{-\frac{\alpha(1-H)(u-v)}{H}}}{\left|1-\mathrm{e}^{-\frac{\alpha(u-v)}{H}}\right|^{2-2H}}
$$

is obviously positive. Thus we conclude that the covariance of the increments of $Y^{(\alpha)}$ is positive and the increments of $Y^{(\alpha)}$ are positively correlated. $\square$

Proposition 3.17 ([29], Prop. 3.8.). *In the case $H \in (\frac{1}{2}, 1)$, the variance of the increments of $Y^{(\alpha)}$ is*

$$\mathbf{E}\left((Y_t^{(\alpha)} - Y_s^{(\alpha)})^2\right) = 2\int_0^{t-s} (t-s-x)k_{\alpha,H}(x)dx. \tag{3.23}$$

Proof. Applying Proposition 3.13, we infer

$$\begin{aligned}\mathbf{E}\left((Y_t^{(\alpha)} - Y_s^{(\alpha)})^2\right) &= \int_s^t \int_s^t r_{\alpha,H}(u,v)dvdu \\ &= 2\int_s^t \int_s^u r_{\alpha,H}(u,v)dvdu,\end{aligned}$$

by Corollary 3.14. Substituting x with u and y by $u-v$ we obtain

$$\begin{aligned}\mathbf{E}\left((Y_t^{(\alpha)} - Y_s^{(\alpha)})^2\right) &= 2\int_s^t \int_0^{x-s} k_{\alpha,H}(y)dydx \\ &= 2\int_0^{t-s} \int_{y+s}^t k_{\alpha,H}(y)dxdy. \end{aligned} \tag{3.24}$$

We are allowed to change the order of integration in (3.24) using the Fubini theorem [54, Theorem 7.8.], since $k_{\alpha,H}$ is positive and continuous. Note that we have to take into account that the limits of the integral are changing, too. We can calculate the inner integral

$$\mathbf{E}\left((Y_t^{(\alpha)} - Y_s^{(\alpha)})^2\right) = 2\int_0^{t-s} k_{\alpha,H}(y)(t-(y+s))dy,$$

verifying the statement. $\square$

The fact that in the next proposition the covariance is also positive, follows from the Corollary 3.16.

Proposition 3.18 ([29], Prop. 3.8.). *In the case $H \in (\frac{1}{2}, 1)$, the covariance of the $Y^{(\alpha)}$ is*

$$\begin{aligned}\mathbf{E}\left(Y_t^{(\alpha)} Y_s^{(\alpha)}\right) &= \int_0^t (t-x)k_{\alpha,H}(x)dx \\ &\quad + \int_0^s (s-x)k_{\alpha,H}(x)dx - \int_0^{t-s} (t-s-x)k_{\alpha,H}(x)dx.\end{aligned} \tag{3.25}$$

Proof. Using identity $ab = \frac{1}{2}\left(a^2 + b^2 - (a-b)^2\right)$, for all $a, b \in \mathbb{R}$, we observe that

$$\mathbf{E}\left(Y_t^{(\alpha)} Y_s^{(\alpha)}\right) = \frac{1}{2}\left(\mathbf{E}\left((Y_t^{(\alpha)})^2\right) + \mathbf{E}\left((Y_s^{(\alpha)})^2\right) - \mathbf{E}\left((Y_t^{(\alpha)} - Y_s^{(\alpha)})^2\right)\right).$$

Now, from Proposition 3.17 we conclude the result as follows

$$\begin{aligned} \mathbf{E}\left(Y_t^{(\alpha)} Y_s^{(\alpha)}\right) &= \int_0^t (t-x) k_{\alpha,H}(x) dx + \int_0^s (s-x) k_{\alpha,H}(x) dx \\ &\quad - \int_0^{t-s} (t-s-x) k_{\alpha,H}(x) dx. \end{aligned}$$

□

In this subsection we consider the covariance and the variance of $Y^{(\alpha)}$ when the time parameter tends towards infinity. We rewrite the symmetric kernel in Corollary 3.14 as follows

$$r_{\alpha,H}(u,v) = k_{\alpha,H}(u-v),$$

where

$$k_{\alpha,H}(x) = C(\alpha, H) \frac{\mathrm{e}^{-\frac{\alpha(1-H)x}{H}}}{\left|1 - \mathrm{e}^{-\frac{\alpha x}{H}}\right|^{2-2H}}. \tag{3.26}$$

Proposition 3.19 ([29], Prop. 3.8.)**.** *In the case $H \in (\frac{1}{2}, 1)$, the variance of the $Y^{(\alpha)}$ satisfies*

$$\mathbf{E}\left((Y_t^{(\alpha)})^2\right) = \mathbf{O}(t) \ \text{as } t \to \infty. \tag{3.27}$$

Proof. Applying Proposition 3.17 and substituting $s = 0$, we obtain

$$\begin{aligned} \mathbf{E}\left((Y_t^{(\alpha)})^2\right) &= 2\int_0^t (t-x) k_{\alpha,H}(x) dx \\ &= 2t \int_0^t k_{\alpha,H}(x) dx - 2\int_0^t x k_{\alpha,H}(x) dx. \end{aligned} \tag{3.28}$$

Using (3.28) we prove that

$$\mathbf{E}\left((Y_t^{(\alpha)})^2\right) = \mathbf{O}(t).$$

For this, it is sufficient to show the properties

$$\int_0^\infty k_{\alpha,H}(x) dx < \infty \tag{3.29}$$

and

$$\int_0^\infty x k_{\alpha,H}(x) dx < \infty. \tag{3.30}$$

We do not need approximate the integral in (3.29) since it is possible to calculate its exact value. Substituting $\mathrm{e}^{-\frac{\alpha x}{H}}$ with u, we have $dx = -\frac{H}{\alpha}\frac{1}{u}du$, and

$$\begin{aligned}
\lim_{t\to\infty}\int_0^t k_{\alpha,H}(x)dx &= \lim_{t\to\infty}\int_{\mathrm{e}^{-\frac{\alpha t}{H}}}^1 C(\alpha,H)\frac{H}{\alpha}u^{-H}(1-u)^{2H-2}du \\
&= C(\alpha,H)\frac{H}{\alpha}\lim_{t\to 0}\int_t^1 u^{-H}(1-u)^{2H-2}du \\
&= C(\alpha,H)\frac{H}{\alpha}\mathrm{Beta}(1-H,2H-1).
\end{aligned}$$

To prove (3.30), we have to study both limits, since there may be difficulties at zero and infinity. We manipulate the kernel

$$\begin{aligned}
xk_{\alpha,H}(x) &= C(\alpha,H)\frac{x\mathrm{e}^{-\frac{\alpha(1-H)x}{H}}}{\left|1-\mathrm{e}^{-\frac{\alpha x}{H}}\right|^{2-2H}} \\
&= C(\alpha,H)\frac{x\mathrm{e}^{\frac{\alpha x(1-H)}{H}}}{\left|\mathrm{e}^{\frac{\alpha x}{H}}-1\right|^{2-2H}}. \qquad (3.31)
\end{aligned}$$

We use the l'Hospitals rule, to obtain

$$\begin{aligned}
\lim_{x\to 0}\frac{x}{\left|\mathrm{e}^{\frac{\alpha x}{H}}-1\right|^{2-2H}} &= \lim_{x\to 0}\frac{1}{(2-2H)\frac{\alpha}{H}\mathrm{e}^{\frac{\alpha x}{H}}\left|\mathrm{e}^{\frac{\alpha x}{H}}-1\right|^{1-2H}} \\
&= \lim_{x\to 0}\frac{H\left|\mathrm{e}^{\frac{\alpha x}{H}}-1\right|^{2H-1}}{(2-2H)\alpha\mathrm{e}^{\frac{\alpha x}{H}}} = 0,
\end{aligned}$$

since $H\in(\frac{1}{2},1)$. Therefore

$$\int_0^\varepsilon xk_{\alpha,H}(x)dx < \infty$$

for any $\varepsilon > 0$.

Thus, we have to study the integral in (3.30) for large values of x. For all $\alpha > 0$ and $H\in(\frac{1}{2},1)$ we can always find n such that

$$1-\mathrm{e}^{\frac{-\alpha x}{H}} > \frac{1}{2},$$

when $x > n$. Let $a > n$, we approximate the integral

$$\int_a^\infty \frac{x\mathrm{e}^{\frac{-\alpha x(1-H)}{H}}}{|1-\mathrm{e}^{-\frac{\alpha x}{H}}|^{2-2H}}dx < \int_a^\infty 2^{2-2H}x\mathrm{e}^{\frac{-\alpha x(1-H)}{H}}dx < \infty$$

and therefore

$$\int_a^\infty xk_{\alpha,H}dx < \infty.$$

We have finally verified that

$$\int_0^\infty k_{\alpha,H}(x)dx < \infty \quad \text{and} \quad \int_0^\infty x k_{\alpha,H}(x)dx < \infty.$$

Since

$$\mathbf{E}\left((Y_t^{(\alpha)})^2\right) < Kt, \text{ for some finite } K,$$

we conclude that

$$\mathbf{E}\left((Y_t^{(\alpha)})^2\right) = \mathbf{O}(t) \text{ as } t \to \infty,$$

thereby completing the proof. □

Corollary 3.20 ([29], Prop. 3.8.)**.** *The covariance of* $Y^{(\alpha)}$ *satisfies*

$$\lim_{t\to\infty} \mathbf{E}\left(Y_t^{(\alpha)} Y_s^{(\alpha)}\right) = s\int_0^\infty k_{\alpha,H}(x)dx + \int_0^s (s-x)k_{\alpha,H}(x)dx. \tag{3.32}$$

Proof. The proof is a straightforward calculation. Applying Proposition 3.18, we obtain

$$\begin{aligned}
\lim_{t\to\infty} \mathbf{E}\left(Y_t^{(\alpha)} Y_s^{(\alpha)}\right) &= \lim_{t\to\infty} \Bigg(\int_0^t (t-x)k_{\alpha,H}(x)dx \\
&\qquad + \int_0^s (s-x)k_{\alpha,H}(x)dx \\
&\qquad - \int_0^{t-s} (t-s-x)k_{\alpha,H}(x)dx \Bigg) \\
&= \int_0^s (s-x)k_{\alpha,H}(x)dx + \lim_{t\to\infty}\Bigg(\int_0^t (t-x)k_{\alpha,H}(x)dx \\
&\qquad - \int_0^{t-s} (t-x)k_{\alpha,H}(x)dx + \int_0^{t-s} s k_{\alpha,H}(x)dx \Bigg),
\end{aligned}$$

where we can combine the integrals yielding

$$\begin{aligned}
\lim_{t\to\infty} \mathbf{E}\left(Y_t^{(\alpha)} Y_s^{(\alpha)}\right) &= \int_0^s (s-x)k_{\alpha,H}(x)dx + \lim_{t\to\infty}\Bigg(\int_{t-s}^t (t-x)k_{\alpha,H}(x)dx \\
&\qquad + \int_0^{t-s} s k_{\alpha,H}(x)dx \Bigg) \\
&= \int_0^s (s-x)k_{\alpha,H}(x)dx + \int_0^\infty s k_{\alpha,H}(x)dx,
\end{aligned}$$

thereby completing the proof. □

3.2.5 Increment process of $Y^{(\alpha)}$

We are ready to define the increment process of $Y^{(\alpha)}$, and prove that it is a short-range dependent stationary process. We already proved in Proposition 3.8 that the process $Y^{(\alpha)}$ itself has stationary increment. Next we prove the same result but in more useful way.

Proposition 3.21 ([29], Cor. 3.7.)**.** *In the case $H \in (\frac{1}{2}, 1)$, the increment process of $Y^{(\alpha)}$*

$$I_Y := \{I_{Y_n} : n = 0, 1, \ldots\} = \{Y^{(\alpha)}_{n+1} - Y^{(\alpha)}_n : n = 0, 1, \ldots\}$$

is stationary for any $\alpha > 0$.

Proof. According to Proposition 3.13, the covariance of the increments of $Y^{(\alpha)}$ is

$$\begin{aligned}
&\mathbf{E}\left(\left(Y^{(\alpha)}_{n+1} - Y^{(\alpha)}_n\right)\left(Y^{(\alpha)}_{m+1} - Y^{(\alpha)}_m\right)\right) \\
&\quad = C(\alpha, H) \int_n^{n+1}\int_m^{m+1} \frac{\mathrm{e}^{-\frac{\alpha(1-H)(u-v)}{H}}}{\left|1 - \mathrm{e}^{-\frac{\alpha(u-v)}{H}}\right|^{2-2H}} dudv.
\end{aligned}$$

To prove the stationarity, we need to show that for any $h > 0$, we have

$$\begin{aligned}
&\mathbf{E}\left(\left(Y^{(\alpha)}_{n+1+h} - Y^{(\alpha)}_{n+h}\right)\left(Y^{(\alpha)}_{m+1+h} - Y^{(\alpha)}_{m+h}\right)\right) \\
&\qquad = \mathbf{E}\left(\left(Y^{(\alpha)}_{n+1} - Y^{(\alpha)}_n\right)\left(Y^{(\alpha)}_{m+1} - Y^{(\alpha)}_m\right)\right).
\end{aligned}$$

In (3.33) we can make the substitutions $s := u - h$ and $r := v - h$

$$\begin{aligned}
&\mathbf{E}\left(\left(Y^{(\alpha)}_{n+1+h} - Y^{(\alpha)}_{n+h}\right)\left(Y^{(\alpha)}_{m+1+h} - Y^{(\alpha)}_{m+h}\right)\right) \\
&\quad = C(\alpha, H) \int_{n+h}^{n+1+h}\int_{m+h}^{m+1+h} \frac{\mathrm{e}^{-\frac{\alpha(1-H)(u-v)}{H}}}{\left|1 - \mathrm{e}^{-\frac{\alpha(u-v)}{H}}\right|^{2-2H}} dudv, \qquad (3.33) \\
&\quad = C(\alpha, H) \int_n^{n+1}\int_m^{m+1} \frac{\mathrm{e}^{-\frac{\alpha(1-H)(s+h-(r+h))}{H}}}{\left|1 - \mathrm{e}^{-\frac{\alpha(s+h-(r+h))}{H}}\right|^{2-2H}} dsdr, \\
&\quad = C(\alpha, H) \int_n^{n+1}\int_m^{m+1} \frac{\mathrm{e}^{-\frac{\alpha(1-H)(s-r))}{H}}}{\left|1 - \mathrm{e}^{-\frac{\alpha(s-r)}{H}}\right|^{2-2H}} dsdr, \\
&\mathbf{E}\left(\left(Y^{(\alpha)}_{n+1} - Y^{(\alpha)}_n\right)\left(Y^{(\alpha)}_{m+1} - Y^{(\alpha)}_m\right)\right).
\end{aligned}$$

Hence the sequence I_Y is stationary according to Theorem 1.3. □

Proposition 3.22 ([29], Cor. 3.7.)**.** *In the case $H \in (\frac{1}{2}, 1)$, the increment process I_Y defined in Proposition 3.21 is short-range dependent.*

Proof. The idea of the proof is to apply Proposition 3.13. We have to show that the condition in Definition 1.18

$$\sum_{n=1}^{\infty} \mathbf{E}(I_{Y_1} I_{Y_n}) = \sum_{n=1}^{\infty} \mathbf{E}\left(Y^{(\alpha)}_1\left(Y^{(\alpha)}_{n+1} - Y^{(\alpha)}_n\right)\right) < \infty$$

holds. We start by considering

$$\begin{aligned}
\mathbf{E}\left(Y_1^{(\alpha)}\left(Y_{n+1}^{(\alpha)}-Y_n^{(\alpha)}\right)\right) &= \int_n^{n+1}\int_0^1 r_{\alpha,H}(u,v)dvdu \\
&= C(\alpha,H)\int_n^{n+1}\int_0^1 \frac{\mathrm{e}^{-\frac{\alpha(1-H)(u-v)}{H}}}{\left|1-\mathrm{e}^{-\frac{\alpha(u-v)}{H}}\right|^{2-2H}}dudv.
\end{aligned}$$

Changing the variable u to $w+n$, we infer that

$$\begin{aligned}
&\mathbf{E}\left(Y_1^{(\alpha)}\left(Y_{n+1}^{(\alpha)}-Y_n^{(\alpha)}\right)\right) \\
&= C(\alpha,H)\mathrm{e}^{-\frac{\alpha(1-H)n}{H}}\int_0^1\int_0^1 \frac{\mathrm{e}^{-\frac{\alpha(1-H)(w-v)}{H}}}{\left|1-\mathrm{e}^{-\frac{\alpha n}{H}}\mathrm{e}^{-\frac{\alpha(w-v)}{H}}\right|^{2-2H}}dwdv.
\end{aligned}$$

We are only interested in the large values of n, and therefore we need evaluate only the limit. To change the order of the integral and the limit we have to use Extended Monotone Convergence Theorem [1, p.47]. Let

$$g_n(w,v) = \frac{\mathrm{e}^{-\frac{\alpha(1-H)(w-v)}{H}}}{\left|1-\mathrm{e}^{-\frac{\alpha n}{H}}\mathrm{e}^{-\frac{\alpha(w-v)}{H}}\right|^{2-2H}}.$$

If the sequence of functions g_n is increasing with respect to n, and we find the integrable minorant, or if the sequence of functions is decreasing with respect to n, and we find the integrable majorant, then we are allowed to change the order of the integration and the limit. We consider that in two separable situations: when $w > v$ and $v \geq w$.

- Let $1 \geq w > v \geq 0$ and $n \geq 2$. Then

  $$1-\mathrm{e}^{-\frac{\alpha n}{H}}\mathrm{e}^{-\frac{\alpha(w-v)}{H}}$$

  is strictly positive for any n and therefore has no poles and is continuous in the bounded area $[0,1]\times[0,1]$. It is also increasing with respect to n and therefore functions has a minorant

  $$1-\mathrm{e}^{-\frac{\alpha n}{H}}\mathrm{e}^{-\frac{\alpha(w-v)}{H}} > 1-\mathrm{e}^{-\frac{2\alpha}{H}}\mathrm{e}^{-\frac{\alpha(w-v)}{H}}.$$

  We approximate the sequence of functions $g_n(w,v)$ to find the majorant of the decreasing sequence of functions $g_n(w,v)$ and obtain

  $$\begin{aligned}
  g_n(w,v) &< \frac{\mathrm{e}^{-\frac{\alpha(1-H)(w-v)}{H}}}{\left(1-\mathrm{e}^{-\frac{2\alpha}{H}}\mathrm{e}^{-\frac{\alpha(w-v)}{H}}\right)^{2-2H}} \\
  &= \frac{1}{\left(\mathrm{e}^{\frac{\alpha(w-v)}{2H}}-\mathrm{e}^{-\frac{\alpha(w-v)}{2H}-\frac{2\alpha}{H}}\right)^{2-2H}},
  \end{aligned}$$

  which is continuous on the bounded area $[0,1]\times[0,1]$ with no poles, therefore it is integrable, since $w-v<2$. Hence we have an integrable majorant and this allows to change the order of operations by Extended Monotone Convergence Theorem.

- Let $0 \leq w \leq v \leq 1$ and $n \geq 2$. Since we are only interested in large values of n, we need only to evaluate the sequence when $n \geq 2$. The sequence of functions

$$\begin{aligned} g_n(w,v) &= \frac{\mathrm{e}^{\frac{\alpha(1-H)(v-w)}{H}}}{\left|1-\mathrm{e}^{-\frac{\alpha n}{H}}\mathrm{e}^{\frac{\alpha(v-w)}{H}}\right|^{2-2H}} \\ &= \frac{\mathrm{e}^{\frac{\alpha(1-H)(v-w)}{H}}}{\left(1-\mathrm{e}^{-\frac{\alpha n}{H}}\mathrm{e}^{\frac{\alpha(v-w)}{H}}\right)^{2-2H}} \end{aligned}$$

since $\alpha n - \alpha(w-v) > 0$. And it is again decreasing with similar reasoning as in the case $w > v$. Also for any $n \geq 2$ we may approximate

$$g_n(w,v) < \frac{\mathrm{e}^{\frac{\alpha(1-H)(v-w)}{H}}}{\left(1-\mathrm{e}^{-\frac{2\alpha}{H}}\mathrm{e}^{\frac{\alpha(v-w)}{H}}\right)^{2-2H}}$$

and also in this case a majorant which is continuous on the bounded area $[0,1]\times[0,1]$ with no poles, therefore it is integrable.

Subsequently we always have the integrable majorant of the sequence of functions and we can change the order of integration and the limit. We obtain

$$\begin{aligned} &\lim_{n\to\infty}\int_0^1\int_0^1 \frac{\mathrm{e}^{-\frac{\alpha(1-H)(w-v)}{H}}}{\left|1-\mathrm{e}^{-\frac{\alpha n}{H}}\mathrm{e}^{-\frac{\alpha(w-v)}{H}}\right|^{2-2H}}dwdv \\ &= \int_0^1\int_0^1 \mathrm{e}^{-\frac{\alpha(1-H)(w-v)}{H}}dwdv \\ &= \int_0^1 \mathrm{e}^{-\frac{\alpha(1-H)w}{H}}dw\int_0^1 \mathrm{e}^{\frac{\alpha(1-H)(v)}{H}}dv = D < \infty, \end{aligned}$$

where D is constant. Hence, we infer that

$$\mathbf{E}\left(Y_1^{(\alpha)}\left(Y_{n+1}^{(\alpha)} - Y_n^{(\alpha)}\right)\right) < DC(\alpha,H)\mathrm{e}^{-\frac{\alpha(1-H)n}{H}}.$$

Consequently, as $n \to \infty$, we have

$$\mathbf{E}\left(Y_1^{(\alpha)}\left(Y_{n+1}^{(\alpha)} - Y_n^{(\alpha)}\right)\right) = \mathbf{O}(\mathrm{e}^{-\frac{\alpha(1-H)n}{H}}).$$

Lastly, we consider the sum

$$\begin{aligned} \sum_{n=1}^{\infty}\mathbf{E}\left(I_{Y_1}I_{Y_n}\right) &= \sum_{n=1}^{\infty}\mathbf{E}\left(Y_1^{(\alpha)}\left(Y_{n+1}^{(\alpha)} - Y_n^{(\alpha)}\right)\right) \\ &= \lim_{N\to\infty}\sum_{n=1}^{N}\mathbf{E}\left(Y_1^{(\alpha)}\left(Y_{n+1}^{(\alpha)} - Y_n^{(\alpha)}\right)\right), \\ &= \lim_{N\to\infty}\left(\mathbf{E}\left(Y_1^{(\alpha)}Y_{N+1}^{(\alpha)}\right)\right) - \mathbf{E}\left(\left(Y_1^{(\alpha)}\right)^2\right). \end{aligned}$$

By Proposition 3.18 and (3.28) of the proof of Proposition 3.19, we obtain

$$\begin{aligned}
&\lim_{N\to\infty} \mathbf{E}\left(Y_1^{(\alpha)}Y_{N+1}^{(\alpha)}\right) - \mathbf{E}\left(\left(Y_1^{(\alpha)}\right)^2\right) \\
&= \int_0^\infty k_{\alpha,H}(x)dx + \int_0^1 k_{\alpha,H}(x)dx - \int_0^1 xk_{\alpha,H}(x)dx \\
&\quad -2\int_0^1 k_{\alpha,H}(x)dx + 2\int_0^1 xk_{\alpha,H}(x)dx \\
&= \int_0^\infty k_{\alpha,H}(x)dx - \int_0^1 k_{\alpha,H}(x)dx + \int_0^1 xk_{\alpha,H}(x)dx < \infty
\end{aligned}$$

due to equations (3.29) and (3.30).

Applying the previous equations, we can conclude

$$\sum_{n=1}^{\infty} \mathbf{E}\left(I_{Y_1} I_{Y_n}\right) = \lim_{N\to\infty} \left(\mathbf{E}\left(Y_1^{(\alpha)}Y_N^{(\alpha)}\right)\right) - \mathbf{E}\left(\left(Y_1^{(\alpha)}\right)^2\right) < \infty,$$

thereby completing the proof. □

3.3 Conclusion of the stationarity of the fOU processes

The following processes are stationary;

- $\{U_t^{(Z,\alpha)} : t \in \mathbb{R}\}$ the fractional Ornstein–Uhlenbeck process of the first kind, fOU(1) (follows from Definition 1.42),
- $\{X_t^{(D,\alpha)} : t \in \mathbb{R}\}$ the Doob transformation of fBm, fOU (follows from Proposition 2.1),
- $\{U_t^{(D,\gamma)} : t \in \mathbb{R}\}$ the fractional Ornstein–Uhlenbeck process of the second kind, fOU(2) (follows from Proposition 3.8),
- $\{I_{Y_n} = n = 0, 1, \ldots\}$ the increment process of $\{Y_t^{(\alpha)} : t \in \mathbb{R}\}$ (follows from Proposition 3.21).

The following processes have stationary increments;

- $\{Z_t : t \geq 0\}$ fractional Brownian motion (follows from Definition 1.35).
- $\{Y^{(\alpha)} : t \in \mathbb{R}\}$ (follows from Proposition 3.8, (however it is not itself stationary)).
- $\{Y^{(1)} : t \in \mathbb{R}\}$ (follows from Proposition 3.8).

3.4 fOU(2) is a short-range dependent for $H > \frac{1}{2}$

We have already calculated long- or short-range dependence properties for all our stationary processes except fOU(2). We prove that fOU(2) is short-range dependent. In this section we consider the rate of decrease of the covariance of $U^{(D,\gamma)}$, where $\gamma > 0$, and after that deduce that the process is short-range dependent.

Proposition 3.23 ([29], Prop. 3.11.). *In the case $H \in (\frac{1}{2}, 1)$, the covariance of the process $\{U_t^{(D,\gamma)} : t \in \mathbb{R}\}$, fOU(2), decreases exponentially*

$$\mathbf{E}\left(U_t^{(D,\gamma)} U_s^{(D,\gamma)}\right) = \mathbf{O}(\mathrm{e}^{-\min\{\gamma, \frac{1-H}{H}\}t}), \ \textit{as } t \to \infty. \tag{3.34}$$

Proof. When we calculate the asymptotic covariance of $U^{(D,\gamma)}$, we may, without loss of generality, set $s = 0$. If $T > 0$, then

$$\begin{aligned}
&\mathbf{E}\left(U_t^{(D,\gamma)} U_0^{(D,\gamma)}\right) \\
&= (2H-1)H^{2H-1}\mathrm{e}^{-\gamma t} \int_{-\infty}^{t}\int_{-\infty}^{0} \frac{\mathrm{e}^{(\gamma-1+\frac{1}{H})(u+v)}}{|\mathrm{e}^{\frac{u}{H}} - \mathrm{e}^{\frac{v}{H}}|^{2-2H}} dvdu \\
&= (2H-1)H^{2H-1}\mathrm{e}^{-\gamma t} \left(\int_{-\infty}^{T}\int_{-\infty}^{0} \frac{\mathrm{e}^{(\gamma-1+\frac{1}{H})(u+v)}}{|\mathrm{e}^{\frac{u}{H}} - \mathrm{e}^{\frac{v}{H}}|^{2-2H}} dvdu \right. \\
&\qquad \left. + \int_{T}^{t}\int_{-\infty}^{0} \frac{\mathrm{e}^{(\gamma-1+\frac{1}{H})(u+v)}}{|\mathrm{e}^{\frac{u}{H}} - \mathrm{e}^{\frac{v}{H}}|^{2-2H}} dvdu \right) \\
&= (2H-1)H^{2H-1}\mathrm{e}^{-\gamma t}\left(D_1(T) + D_2(t)\right).
\end{aligned}$$

Since $D_1(T)$ does not depend on t, we obtain

$$\lim_{t\to\infty} (2H-1)H^{2H-1}\mathrm{e}^{-\gamma t} D_1(T) = 0.$$

Thus

$$(2H-1)H^{2H-1}\mathrm{e}^{-\gamma t} D_1(T) = \mathbf{O}(\mathrm{e}^{-\gamma t}), \ \text{as } t \to \infty. \tag{3.35}$$

For the term $D_2(t)$, we factorise the denominator and deduce

$$\begin{aligned}
D_2(t) &= \int_{T}^{t}\int_{-\infty}^{0} \frac{\mathrm{e}^{(\gamma-1+\frac{1}{H})(u+v)}}{\mathrm{e}^{2u(\frac{1}{H}-1)}|1 - \mathrm{e}^{\frac{v-u}{H}}|^{2-2H}} dvdu \\
&= \int_{T}^{t}\int_{-\infty}^{0} \frac{\mathrm{e}^{(\gamma-1+\frac{1}{H})u+(\gamma+1-\frac{1}{H})v}}{|1 - \mathrm{e}^{\frac{v-u}{H}}|^{2-2H}} dvdu.
\end{aligned}$$

In the previous integral the pair (u, v) belongs to the set $(T, t) \times (-\infty, 0)$, where the difference $v - u$ is always less than or equal to $-T$, which implies

$$1 \geq \left(1 - \mathrm{e}^{\frac{v-u}{H}}\right)^{2(1-H)} \geq \left(1 - \mathrm{e}^{\frac{-T}{H}}\right)^{2(1-H)}.$$

We deduce further

$$\begin{aligned}
&\int_T^t \int_{-\infty}^0 \frac{\mathrm{e}^{(\gamma+1-\frac{1}{H})u+(\gamma-1+\frac{1}{H})v}}{\left(1-\mathrm{e}^{\frac{v-u}{H}}\right)^{2-2H}} dvdu \\
&\quad \leq \int_T^t \int_{-\infty}^0 \frac{\mathrm{e}^{(\gamma+1-\frac{1}{H})u+(\gamma-1+\frac{1}{H})v}}{\left(1-\mathrm{e}^{\frac{-T}{H}}\right)^{2-2H}} dvdu \\
&\quad = (1-\mathrm{e}^{\frac{-T}{H}})^{-2+2H} \int_T^t \mathrm{e}^{(\gamma+1-\frac{1}{H})u} du \int_{-\infty}^0 \mathrm{e}^{(\gamma-1+\frac{1}{H})v} dv \\
&\quad = (1-\mathrm{e}^{\frac{-T}{H}})^{-2+2H} \left(\frac{\mathrm{e}^{(\gamma+1-\frac{1}{H})t}-\mathrm{e}^{(\gamma+1-\frac{1}{H})T}}{\gamma+1-\frac{1}{H}}\right)\left(\frac{1}{\gamma-1+\frac{1}{H}}\right).
\end{aligned}$$

Approximating, we obtain

$$\begin{aligned}
&(2H-1)H^{2H-1}\mathrm{e}^{-\gamma t}D_2(t) \\
&\quad \leq (2H-1)H^{2H-1}\mathrm{e}^{-\gamma t}(1-\mathrm{e}^{\frac{-T}{H}})^{-2+2H} \\
&\qquad\qquad \cdot \left(\frac{\mathrm{e}^{(\gamma+1-\frac{1}{H})t}-\mathrm{e}^{(\gamma+1-\frac{1}{H})T}}{\gamma+1-\frac{1}{H}}\right)\left(\frac{1}{\gamma-1+\frac{1}{H}}\right) \\
&\quad \leq C\mathrm{e}^{-\gamma t+(\gamma+1-\frac{1}{H})t} + M_2\mathrm{e}^{-\gamma t} \\
&\quad = C\mathrm{e}^{-\frac{1-H}{H}t} + M_2\mathrm{e}^{-\gamma t}
\end{aligned}$$

and therefore

$$(2H-1)H^{2H-1}\mathrm{e}^{-\gamma t}D_2(t) = \mathbf{O}(\max(\mathrm{e}^{-\frac{1-H}{H}t}, \mathrm{e}^{-\gamma t})), \text{ as } t \to \infty. \tag{3.36}$$

Hence combining (3.35) and (3.36), we deduce

$$\begin{aligned}
\mathbf{E}\left(U_t^{(D,\gamma)}U_s^{(D,\gamma)}\right) &= \mathbf{O}(\max(\mathrm{e}^{-\gamma t}, \mathrm{e}^{-\frac{1-H}{H}t}) \\
&= \mathbf{O}(\mathrm{e}^{\max(-\gamma t, -\frac{1-H}{H}t)}) \\
&= \mathbf{O}(\mathrm{e}^{-\min(\gamma, \frac{1-H}{H})t}), \text{ as } t \to \infty,
\end{aligned}$$

which is the assertion. □

Theorem 3.24 ([29], Prop. 3.11.)**.** *The stationary process $U^{(D,\gamma)}$ is short-range dependent.*

Proof. We see that in the previous Proposition 3.23 the leading term of the covariance of the stationary process $U^{(D,\gamma)}$ is $\mathrm{e}^{-\min(\gamma,\frac{1-H}{H})t}$. With that information and Definition 1.18 we prove that $U^{(D,\gamma)}$ is short-range dependent as follows:

$$\begin{aligned}
\sum_{n=0}^{\infty} \rho_{U^{(D,\gamma)}}(n) &= \sum_{n=0}^{\infty} \mathbf{E}\left(U_n^{(D,\gamma)}U_0^{(D,\gamma)}\right) \\
&\leq C\sum_{n=0}^{\infty} \mathrm{e}^{-\min(\gamma,\frac{1-H}{H})n}
\end{aligned}$$

where the sum converges, since $\frac{1-H}{H} > 0$ and $\gamma > 0$. □

4 Weak convergence

4.1 Weak convergence

According to Proposition 3.19, the growth of the variance of $Y_t^{(\alpha)}$ is asymptotically linear as $t \to \infty$. Recall that for the standard Brownian motion $\{B_t : t \geq 0\}$ the variance of B_t is equal to t. This property - similarity in variances - can be taken further. We prove in this section that $Y^{(\alpha)}$, if the time parameter is scaled properly, converges weakly to a standard Brownian motion.

4.1.1 Weak convergence and tightness

An example of convergence of finite dimensional distributions helps us to understand weak convergence. The following example is taken from Billingsley [5]

Example 4.1. Let $\Omega = [0,1]$, $\mathcal{B}$ be the collection of the Borel sets in $[0,1]$ and $\mathbf{P}$ the Lebesgue measure on $\mathcal{B}$. Then the triplet $(\Omega, \mathcal{B}, \mathbf{P})$ is a probability space. If we define

$$X_t(\omega) = 0$$

and

$$Y_t(\omega) = \begin{cases} 0, & \text{if } t \neq \omega \\ 1, & \text{if } t = \omega, \end{cases}$$

when $0 \leq t \leq 1$ and $\omega \in \Omega$, then for all $0 \leq t \leq 1$

$$\mathbf{P}(X_t = 0) = \mathbf{P}(Y_t = 0) = 1$$

and the stochastic processes $\{X_t : 0 \leq t \leq 1\}$ and $\{Y_t : 0 \leq t \leq 1\}$ have the same finite dimensional distributions. This means that

$$\mathbf{P}(X_{t_1} \leq x_1, \ldots, X_{t_k} \leq x_k) = \mathbf{P}(Y_{t_1} \leq x_1, \ldots, Y_{t_k} \leq x_k)$$

for any choices of x_j and t_i. Note that the finite dimensional distributions are the same, but the the process behaves very differently. In fact,

$$\sup_{0 \leq t \leq 1} X_t(\omega) = 0 \text{ and } \sup_{0 \leq t \leq 1} Y_t(\omega) = 1,$$

for all ω.

Hence it is also true that the convergence of finite dimensional distributions does not imply the convergence of the distribution of every functional of the process. The theory of weak convergence of probability measures in metric spaces gives conditions which guarantee the convergence.

The previous example is a good motivator to study these problems. We now briefly present some basic elements from the theory of weak convergence using [5, p. 5-6, p. 9-10]).

Let

- S be a separable complete metric space,
- $C(S)$ be the class of bounded, continuous real-valued functions on S,
- $\mathcal{S}$ be the σ-algebra of Borel sets of S,
- P be a probability measure on S.

Definition 4.2. Let $\{P_n\}_{n\geq 1}$ be a sequence of probability measures on $(S, \mathcal{S})$. Then, we say P_n *converges weakly to* P, denoted by $P_n \overset{w}{\Rightarrow} P$ if

$$\lim_{n\to\infty} \int_S f dP_n = \int_S f dP$$

for all functions f in $C(S)$.

The classical setting of the weak convergence is $S = \mathbb{R}$ and $\mathcal{S} = \mathcal{B}$, i.e. Borel sets on the line $\mathbb{R}$. In this situation the probability measure $\mathbf{P}$ is completely determined by its distribution function F, which is defined by $F(x) = P((-\infty, x])$, since F has to be continuous from the right. Suppose that $\{P_n\}$ is a sequence of probability measures on $(\mathbb{R}, \mathcal{B})$, with the distribution functions F_n. Then we have

Theorem 4.3 ([5], Th. 2.3.)**.** *Following statements*

(i) $P_n \overset{w}{\Rightarrow} P$*,*

(ii) $F_n(x) \to F(x)$ *for all continuity points* x *of* F

are equivalent.

In a more general setting, we have the probability space $(\Omega, \mathcal{B}, \mathbf{P})$ and $X : \Omega \to \mathbb{R}$ a real-valued random variable. Let P_X denote the distribution of X, i.e.

$$P_X(A) = \mathbf{P}X^{-1}(A) = \mathbf{P}(\{\omega : X(\omega) \in A\}) = \mathbf{P}(\{X \in A\}),$$

where $A \in \mathcal{B}$. Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables defined on the probability spaces $(\Omega_n, \mathcal{B}_n, \mathbf{P}_n)$ with distributions $(P_X)_n = \mathbf{P}_n X^{-1}$, in other words $P_n(x) = \mathbf{P}_n(X_n \leq x)$.

Definition 4.4. If $P_n \overset{w}{\Rightarrow} P$, then we say that the sequence $\{X_n\}_{n\geq 1}$ *converges in distribution* to X and we write $X_n \overset{d}{\Rightarrow} X$.

Definition 4.5. A family Π of probability measures on S is said to be *relatively compact* if each sequence $\{P_n\}_{n\geq 1}$ of the elements of the family Π contains some subsequence $\{P_{n_i}\}_{n\geq 1}$ converging weakly to some probability measure P.

The Prokhorov Theorem, for example in [5], combines these basic elements of weak convergence and probability theory. We recall that a family Π of probability measures is *tight* if for every ε there exists a compact set K_ε such that $P(K_\varepsilon) > 1 - \varepsilon$ for every P in Π, see, for example, Billingsley [6, pages 8 and 58−59].

Theorem 4.6 ([5], Th. 4.1.)**.** *Family* Π *of probability measures is relatively compact if and only if it is tight.*

From Theorem 4.6, we obtain

Corollary 4.7. *If a family of probability measures* $\{P_n\}_{n\geq 1}$ *is tight, then it is relatively compact and it has a subsequence converging weakly to* P*.*

The same subject matter was also studied by Lamperti and he proved an important result [34]. We use our terminology in this theory, but first define the Lip_α space. Let Lip_α be the space of all real-valued functions $t \mapsto X_t$ defined for $t \in [0,1]$, with $X_0 = 0$ and such that

$$\|X\|_\alpha = \sup_{t_1,t_2\in[0,1]} \frac{|X_{t_2} - X_{t_1}|}{|t_2 - t_1|^\alpha} + \max_{t_2\in[0,1]} |X_{t_2}| < \infty.$$

Lamperti presented his theorems of continuity, which we need in using separable stochastic or Gaussian processes. Therefore we have to represent one definition of separability, see, for example, Creamer and Leadbetter [14]

Definition 4.8 ([14] p.48)**.** A stochastic process $\{X_t : t \in [0,t]\}$ is said to be *separable* if there is a countable subset S of $[0,1]$ such that for any open interval $I \subset [0,1]$, with probability one,

$$\sup_{t\in I\cap S} X_t = \sup_{t\in I} X_t \text{ and } \inf_{t\in I\cap S} X_t = \inf_{t\in I} X_t.$$

Theorem 4.9 ([34], p.432)**.** *Let* $\{X_t : t \in [0,t]\}$ *and* $X_0 = 0$ *be a sequence of separable stochastic processes satisfying the Kolmogorov criterion (Th. 1.10) with* α*,* β *and* M *independent on* n*. Suppose also that the finite dimensional distributions of* $\{X_t^{(n)} : t \in [0,1]\}$ *converge when* $n \to \infty$*. Then there exists a process* $\{X_t : t \in [0,t]\}$ *whose finite dimensional distributions are these limits whose path-functions belong a.s. to* Lip_γ *for every* $\gamma < \dfrac{\beta}{\alpha}$ *and such that*

$$\lim_{n\to\infty} \int_S f dP_n = \int_S f dP$$

for every functional f *which is continuous at almost all points of* Lip_γ *for some* $\gamma < \dfrac{\beta}{\alpha}$ *(with respect to the measure induced by the process*

$$\{X_t : t \in [0,t]\}$$

)

In the case of Gaussian processes we have a more specific Theorem (with our terminology).

Theorem 4.10 ([34], Cor. 2). *Let $\{X_t^{(n)} : t \in [0,1]\}$ and $n = 1, 2, \ldots$ be a sequence of real separable Gaussian stochastic processes such that*

$$\mathbf{E}\left(X_t^{(n)}\right) = \mu_n(t)$$
$$\mathbf{E}\left(\left(X_s^{(n)} - \mu_n(s)\right)\left(X_t^{(n)} - \mu_n(t)\right)\right) = \rho_n(s,t).$$

We assume also that

$$\lim_{n\to\infty} \mu_n(t) = \mu(t) \text{ and } \lim_{n\to\infty} \rho_n(s,t) = \rho(s,t).$$

Suppose also that there exist constants $\xi \in [0,2]$ and $A, B < \infty$ such that for t, $t + \Delta t \in [0,1]$

$$|\mu_n(t+\Delta t) - \mu_n(t)| \le A|\Delta t|^{\frac{\xi}{2}} \text{ and}$$
$$|\rho_n(t,t) - 2\rho_n(t,t+\Delta t) + \rho_n(t+\Delta t, t+\Delta t)| \le B|\Delta t|^{\xi}.$$

Then there is a separable Gaussian process $\{X_t : t \in [0,t]\}$ with the mean function $\mu(t)$, the covariance $\rho(s,t)$ and whose paths belongs a.s. to Lip_γ for every $\gamma < \frac{\xi}{2}$ such that

$$\lim_{n\to\infty} \int_S f dP_n = \int_S f dP$$

holds for every functional f which is continuous a.s. in the topology of Lip_γ (with respect to the measure induced by the process $\{X_t : t \in [0,t]\}$) for some $\gamma < \frac{\xi}{2}$.

In the next section we will consider the weak convergence of

$$Y_t^{(\alpha)} = \int_0^t \mathrm{e}^{-\alpha} s dZ_{\tau_s},$$

where $\tau_t = \dfrac{H\mathrm{e}^{\frac{\alpha t}{H}}}{\alpha}$. We recall that sample paths of $Y^{(\alpha)}$ are locally Hölder continuous of any order $\beta < H$ by Proposition 3.10, and therefore the process $Y^{(\alpha)}$ have a continuous modification. This guarantees that the process is also separable and we may apply Theorem 4.10. In this theorem there is a condition for tightness

$$|\rho_n(t,t) - 2\rho_n(t,t+\Delta t) + \rho_n(t+\Delta t, t+\Delta t)| \le B|\Delta t|^{\xi}.$$

We rewrite the left-hand side of the previous inequality

$$\begin{aligned}
&\rho_n(t,t) - 2\rho_n(t,t+\Delta t) + \rho_n(t+\Delta t, t+\Delta t)\\
&\quad = \mathbf{E}(X_t^{(n)} - \mu_n(t))^2 - 2\mathbf{E}(X_t^{(n)} - \mu_n(t))(X_{t+\Delta t}^{(n)} - \mu_n(t+\Delta t))\\
&\qquad + \mathbf{E}(X_{t+\Delta t}^{(n)} - \mu_n(t+\Delta t))^2\\
&\quad = \mathbf{E}\left((X_t^{(n)} - \mu_n(t)) - (X_{t+\Delta t}^{(n)} - \mu_n(t+\Delta t))\right)^2
\end{aligned}$$

and in our case, where processes are Gaussian processes of zero mean, we obtain

$$\begin{aligned}
&|\rho_n(t,t) - 2\rho_n(t,t+\Delta t) + \rho_n(t+\Delta t, t+\Delta t)|\\
&\quad = \left|\mathbf{E}\left(X_t^{(n)} - X_{t+\Delta t}^{(n)}\right)^2\right|.
\end{aligned}$$

Hence we may conclude that weak convergence of a zero mean locally Hölder continuous Gaussian process is tight, if we can prove that the variance of increments of the Gaussian process is bounded by $B|\Delta t|^{\xi}$, where Δt is increment of the time parameter and $\xi \in [0,2]$.

4.2 Weak convergence of $Y^{(\alpha)}$

In this section, we prove that the process $\{Z_t^{(a,\alpha)} : t \geq 0\} := \{\frac{1}{\sqrt{a}} Y_{at}^{(\alpha)},\, t \geq 0\}$ converges weakly to the scaled Brownian motion. This holds if we show that the finite dimensional distributions of this process converge weakly as $a \to \infty$ and that the sequence of probability measures is tight. This is the contents of the next result.

In the next theorem we state the precise result for arbitrary $\alpha > 0$.

Proposition 4.11 ([29], Prop. 3.12.)**.** *Let for $t \geq 0$*

$$Z_t^{(a,\alpha)} := \frac{1}{\sqrt{a}} Y_{at}^{(\alpha)},$$

where $a > 0$. If $B = \{B_t : t \geq 0\}$ is a standard Brownian motion starting from 0, then it holds

$$\{Z_t^{(a,\alpha)} : t \geq 0\} \overset{w}{\Rightarrow} \{\sigma B_t : t \geq 0\}, \text{ as } a \to \infty,$$

where $\overset{w}{\Rightarrow}$ stands for the weak convergence in the space of continuous functions and $\sigma(\alpha, H) = \sigma$ is a non-random quantity depending only on $\alpha > 0$ and $H \in (\frac{1}{2}, 1)$.

Proof. First we prove that the finite dimensional distributions of $\{Z_t^{(a,\alpha)} : t \geq 0\}$ converge to the finite dimensional distributions of that $\{\sigma B_t : t \geq 0\}$. Since both processes $Z^{(a,\alpha)}$ and σB are Gaussian processes of zero mean the covariance functions determine their distributions uniquely. Hence, it suffices to verify the convergence of the covariance functions. Applying Proposition 3.18 we infer, for $t > s$

$$\begin{aligned}
\mathbf{E}\left(Z_t^{(a,\alpha)} Z_s^{(a,\alpha)}\right) &= \frac{1}{a}\mathbf{E}\left(Y_{at}^{(\alpha)} Y_{as}^{(\alpha)}\right) \\
&= \frac{1}{a}\left(\int_0^{at} (at - x) k_{\alpha,H}(x) dx + \int_0^{as} (as - x) k_{\alpha,H}(x) dx \right. \\
&\qquad \left. - \int_0^{a(t-s)} (at - as - x) k_{\alpha,H}(x) dx \right),
\end{aligned}$$

where the kernel

$$\begin{aligned}
k_{\alpha,H}(x) &= C(\alpha, H) \frac{\mathrm{e}^{-\frac{\alpha(1-H)x}{H}}}{\left|1 - \mathrm{e}^{-\frac{\alpha x}{H}}\right|^{2-2H}} \\
&= H(2H-1)\left(\frac{\alpha}{H}\right)^{2-2H} \frac{\mathrm{e}^{-\frac{\alpha(1-H)x}{H}}}{\left|1 - \mathrm{e}^{-\frac{\alpha x}{H}}\right|^{2-2H}}
\end{aligned}$$

is as in Proposition 3.13. If $a \to \infty$ we obtain

$$\begin{aligned}
&\lim_{a\to\infty} \mathbf{E}\left(Z_t^{(a,\alpha)} Z_s^{(a,\alpha)}\right) \\
&= \lim_{a\to\infty} \left(\frac{1}{a}\left(\int_0^{at} (at - x) k_{\alpha,H}(x) dx + \int_0^{as} (as - x) k_{\alpha,H}(x) dx \right.\right. \\
&\qquad \left.\left. - \int_0^{a(t-s)} (at - as - x) k_{\alpha,H}(x) dx \right)\right).
\end{aligned}$$

Reorganizing the terms, we infer

$$\begin{aligned}
&\lim_{a\to\infty}\mathbf{E}\left(Z_t^{(a,\alpha)}Z_s^{(a,\alpha)}\right)\\
&=\lim_{a\to\infty}\Bigg(\Bigg(\frac{1}{a}\Bigg[a\int_0^{at}tk_{\alpha,H}(x)dx+a\int_0^{as}sk_{\alpha,H}(x)dx\\
&\qquad -a\int_0^{a(t-s)}tk_{\alpha,H}(x)dx+a\int_0^{a(t-s)}sk_{\alpha,H}(x)dx\Bigg]\\
&\qquad -\frac{1}{a}\Bigg(\int_0^{at}xk_{\alpha,H}(x)dx+\int_0^{as}xk_{\alpha,H}(x)dx-\int_0^{a(t-s)}xk_{\alpha,H}(x)dx\Bigg)\Bigg)\Bigg).
\end{aligned} \tag{4.1}$$

Since we have proved in (3.30) that

$$\int_0^\infty xk_{\alpha,H}(x)dx<\infty,$$

and $t-s>0$, the expression on the last line of (4.1) tends to zero as $a\to\infty$. Hence, we obtain

$$\lim_{a\to\infty}\mathbf{E}\left(Z_t^{(a,\alpha)}Z_s^{(a,\alpha)}\right)=2\int_0^\infty sk_{\alpha,H}(x)dx.$$

After changing the variables in the kernel we conclude for $t>s$

$$\begin{aligned}
&\lim_{a\to\infty}\mathbf{E}\left(Z_t^{(a,\alpha)}Z_s^{(a,\alpha)}\right)\\
&=2H(2H-1)\left(\frac{\alpha}{H}\right)^{1-2H}s\int_0^1\frac{u^{-H}}{|1-u|^{2-2H}}du\\
&=2H(2H-1)\left(\frac{\alpha}{H}\right)^{1-2H}\mathrm{Beta}(1-H,2H-1)s\\
&=2H(2H-1)\left(\frac{\alpha}{H}\right)^{1-2H}\mathrm{Beta}(1-H,2H-1)\min(s,t)\\
&=2H(2H-1)\left(\frac{\alpha}{H}\right)^{1-2H}\mathrm{Beta}(1-H,2H-1)\mathbf{E}(B_tB_s),
\end{aligned}$$

since $\mathbf{E}(B_tB_s)=\min(t,s)$.We define

$$\begin{aligned}
\kappa(\alpha,H)\qquad &:=2H(2H-1)\left(\frac{\alpha}{H}\right)^{1-2H}\mathrm{Beta}(1-H,2H-1)\\
&=2C(\alpha,H)\left(\frac{\alpha}{H}\right)\mathrm{Beta}(1-H,2H-1).
\end{aligned}$$

We have now proved that the finite dimensional distributions of $Z^{(a,\alpha)}$ converge to the finite dimensional distributions of σB with

$$\sigma=\sigma(\alpha,H)=\sqrt{\kappa(\alpha,H)}.$$

We still have to prove the tightness. It suffices to verify, see [34] or Theorem 4.10 that for every $\alpha > 0$ and $H \in (\frac{1}{2}, 1)$ there exists a constant $D(\alpha, H)$ (not depending on a) such that for $t > s$

$$\Delta := \mathbf{E}\left(\left(Z_t^{(a,\alpha)} - Z_s^{(a,\alpha)}\right)^2\right) \le D(\alpha, H)(t-s).$$

Applying Proposition 3.17 we obtain

$$\begin{aligned}
\Delta &= \frac{1}{a}\mathbf{E}\left(\left(Y_t^{(\alpha)} - Y_s^{(\alpha)}\right)^2\right) \\
&= 2\frac{1}{a}\int_0^{a(t-s)} (a(t-s) - x)k_{\alpha,H}(x)dx \\
&= 2\int_0^{a(t-s)} (t-s)k_{\alpha,H}(x)dx - 2\frac{1}{a}\int_0^{a(t-s)} xk_{\alpha,H}(x)dx \\
&\le 2(t-s)\int_0^{\infty} k_{\alpha,H}(x)dx \\
&= D(\alpha, H)(t-s),
\end{aligned}$$

thereby completing the proof. □

5 Conclusion

5.1 Main results

The main result of the dissertation is that we have represented the Doob transformation of fBm via the solution of a Langevin stochastic differential equation, see Proposition 3.2

$$dX_t^{(D,\alpha)} = -\alpha X_t^{(D,\alpha)} dt + dY_t^{(\alpha)},$$

and have analysed the driving process

$$Y_t^{(\alpha)} := \int_0^t \mathrm{e}^{-\alpha s} dZ_{\frac{H\mathrm{e}^{\frac{\alpha s}{H}}}{\alpha}}.$$

Moreover,

$$\{\alpha^H Y_{\frac{t}{\alpha}}^{(\alpha)} : t \geq 0\} \stackrel{d}{=} \{Y_t^{(1)} : t \geq 0\}$$

holds, which was proved in Proposition 3.3. Using this connection and the two-sided extension of $Y_s^{(1)}$ we have defined a new family of processes

$$U_t^{(D,\gamma)} = \mathrm{e}^{-\gamma t} \int_{-\infty}^t \mathrm{e}^{\gamma s} d\widehat{Y}_s^{(1)},$$

where $\gamma > 0$, and named it the fractional Ornstein−Uhlenbeck process of the second kind, fOU(2). See Section 3.1. We have also studied many properties of $U^{(D,\gamma)}$, for example, it is stationary (Proposition 3.8) and its paths are locally Hölder continuous of any order $\beta < H$ (Proposition 3.11).

We have proved that the Doob transformation of fBm $\{X^{D,\alpha} : t \geq 0\}$ is short-range dependent for all $H \in (0,1)$ (in Theorem 2.8). Theorem 2.5 states that the stationary fractional Ornstein−Uhlenbeck process of the first kind is long-range dependent if $H > \frac{1}{2}$ and short-range dependent if $H < \frac{1}{2}$. Therefore we have deduced and proved in Corollary 2.6 that these processes do not have the same finite dimensional distributions (Chapter 2).

We found using the Bochner theorem that (Theorem 2.8) the spectral density function of the Doob transformation of fBm is

$$\Delta'(\gamma) = \frac{1}{2}\left(\frac{H}{\alpha}\right)^{2H}\left(\frac{\alpha}{\pi}\frac{1}{\gamma^2+\alpha^2} - \frac{\alpha}{\pi}\sum_{n=1}^{\infty}\binom{2H}{n}\frac{(-1)^n\left(\frac{n}{H}-1\right)}{\gamma^2+\left(\frac{\alpha n}{H}-\alpha\right)^2}\right).$$

This function is useful, for example, for prediction.

One quite powerful result, when $H \in (\frac{1}{2}, 1)$, is the kernel representation of the covariance function of the increments of $Y^{(\alpha)}$ in Proposition 3.13, namely

$$\begin{aligned}\mathbf{E}\left(\left(Y_{t_2}^{(\alpha)} - Y_{t_1}^{(\alpha)}\right)\left(Y_{s_2}^{(\alpha)} - Y_{s_1}^{(\alpha)}\right)\right) \\ = C(\alpha, H) \int_{t_1}^{t_2} \int_{s_1}^{s_2} \frac{\mathrm{e}^{-\frac{\alpha(1-H)(u-v)}{H}}}{\left|1 - \mathrm{e}^{-\frac{\alpha(u-v)}{H}}\right|^{2-2H}} dudv,\end{aligned}$$

where $s_1 < s_2$, $t_1 < t_2$ and $C(\alpha, H) := H(2H-1)\left(\frac{\alpha}{H}\right)^{2-2H}$. We have generated a lot of information on processes $Y^{(\alpha)}$ and $U^{(D,\gamma)}$ using that kernel, for example, the covariance, the variance of the increments and how they behave when t tends to infinity. We would like to highlight a few results of processes $Y^{(\alpha)}$ likewise $U^{(D,\gamma)}$. The increment process of $Y^{(\alpha)}$ is short-range dependent (Proposition 3.22) and also the process $U^{(D,\gamma)}$ is (Theorem 3.24). In the case $H \in (\frac{1}{2}, 1)$ it holds that fBm and fOU(1) are long-range dependent processes, but fOU(2) is short-range dependent. This offers interesting opportunities to model real life applications with tractable fractional processes.

Finally, we prove the following on weak convergence result:

$$\{\frac{1}{\sqrt{a}} Y_{at}^{(\alpha)} : t \geq 0\} \overset{w}{\Rightarrow} \{\sigma B_t : t \geq 0\},$$

as $a \to \infty$ and $\sigma(\alpha, H) = \sigma$ is a non-random quantity depending only on α and H (Proposition 4.11).

5.2 On some statistical studies of fOU(2)

We briefly present some results of the fractional Ornstein−Uhlenbeck of the second kind defined by us. Azmoodeh and Morlanes [2] and Azmoodeh and Viitasaari [3] have obtained statistically parameter estimation in fOU(2) model. Let $U^{(D,\gamma)} = \{U_t^{(D,\gamma)} : t \in \mathbb{R}\}$ be the non-stationary fractional Ornstein-Uhlenbeck process of the second kind. We have defined that in (3.5) but in the publications [2] and [3] is with initial value $U_0^{(D,\gamma)} = 0$ and drift parameter $\gamma > 0$. Then

$$U_t^{(D,\gamma)} = \int_0^t \mathrm{e}^{-\gamma(t-s)} dY_s^{(1)}.$$

We also need the definition of least squares estimator

$$\widehat{\gamma}_T = -\frac{\int_0^T U_t^{(D,\gamma)} \delta U_t^{(D,\gamma)}}{\int_0^T {U_t^{(D,\gamma)}}^2 dt}, \tag{5.1}$$

where the integral $\int_0^T U_t^{(D,\gamma)} \delta U_t^{(D,\gamma)}$ is the Skorokhod integral. See, for example, Di Nunno, Øksendal and Proske [15].

Theorem 5.1 ([2], Th. 3.1)**.** *The least squares estimator $\widehat{\gamma}_T$ given by (5.1) is weakly consistent, i.e.*

$$\widehat{\gamma}_T \to \gamma$$

in probability, as T tends to infinity.

Theorem 5.2 ([3], Th. 3.1.). *Let $U^{(D,\gamma)}$ be a fractional Ornstein-Uhlenbeck process of the second kind given in definition 3.6. Then as T tends to infinity, we have*

$$\sqrt{T}\left(\frac{1}{T}\int_0^T \left(U_t^{(D,\gamma)}\right)^2 dt - \Psi(\gamma)\right) \to \mathbf{N}(0,\sigma^2),$$

where

$$\Psi(\gamma) := \frac{(2H-1)H^{2H}}{\gamma} Beta((\gamma-1)H+1, 2H-1), \tag{5.2}$$

and where the variance σ^2 is given by

$$\sigma^2 = \frac{2\alpha_H^2 H^{4H-2}}{\gamma^2}\int_{[0,\infty]^3}\Big[\mathrm{e}^{-\gamma x-\gamma|y-z|}\mathrm{e}^{(1-\frac{1}{H})(x+y+z)} \tag{5.3}$$

$$\times\left(1-\mathrm{e}^{-\frac{y}{H}}\right)^{2H-2}\left|\mathrm{e}^{-\frac{x}{H}}-\mathrm{e}^{-\frac{z}{H}}\right|^{2H-2}\Big] dzdxdy \tag{5.4}$$

and $\alpha_H = H(2H-1)$.

Theorem 5.3 ([3], Th. 3.2.). *Assume we observe $U_t^{(D,\gamma)}$ at discrete points $\{t_k = k\Delta_N, k = 0,\dots,N\}$ and $T_N = N\Delta_N$. Assume we have $\Delta_N \to 0, T_N \to \infty$ and $N\Delta_N^2 \to 0$ as N tends to infinity. Set*

$$\widehat{\mu}_{2,N} = \frac{1}{T_N}\sum_{k=1}^N {U_{t_k}^{(D,\gamma)}}^2 \Delta t_k \ \text{ and } \ \widehat{\gamma}_N := \Psi^{-1}(\widehat{\mu}_{2,N}),$$

where Ψ^{-1} is the inverse of function Ψ given in (5.2).

Then $\widehat{\gamma}$ is a strongly consistent estimator of drift parameter γ in the sense that as N tends to infinity, we have

$$\widehat{\gamma}_N \to \gamma$$

almost surely. Moreover, as N tends to infinity, we have

$$\sqrt{T_N}(\widehat{\gamma}_N - \gamma) \xrightarrow{d} \mathcal{N}(0,\sigma^2),$$

where

$$\sigma_\gamma^2 = \frac{\sigma^2}{[\Psi'(\gamma)]^2}$$

and σ is given by (5.3).

A Summary of some properties and definitions

A.1 Gaussian processes: conclusions

We have actually studied five different Gaussian processes. Since some properties are at the beginning of this dissertation and some in the middle part and so on, we present a summary of definitions and some basic properties of these five processes.

fBm	Def.1.27
$\mathbf{E}(Z_t Z_s) = \frac{1}{2}(t^{2H} + s^{2H} - \|t-s\|^{2H}), t, s \geq 0.$	Def.1.27
H- self-similar	Th. 1.28
Locally Hölder continuous of any order $\beta < H$	Th. 1.32
Short-range dependent, if $H < \frac{1}{2}$	Prop. 1.37
Long-range dependent, if $H > \frac{1}{2}$	Prop. 1.37
fOU	Def. 1.43
$\mathbf{E}\left(X_t^{(D,\alpha)} X_s^{(D,\alpha)}\right) = \frac{1}{2}\left(\frac{H}{\alpha}\right)^{2H}\left(\mathrm{e}^{\alpha(t-s)} + \mathrm{e}^{-\alpha(t-s)} - \mathrm{e}^{\alpha(t-s)}\left(1-\mathrm{e}^{-\frac{\alpha(t-s)}{H}}\right)^{2H}\right)$	Prop.2.1
$\mathbf{E}\left(X_t^{(D,\alpha)} X_0^{(D,\alpha)}\right) = \mathbf{O}(\mathrm{e}^{-\alpha t}),\ \text{if } H < \frac{1}{2}$	Cor. 2.4
$\mathbf{E}\left(X_t^{(D,\alpha)} X_0^{(D,\alpha)}\right) = \mathbf{O}(\mathrm{e}^{-\alpha t(\frac{1}{H}-1))}),\ \text{if } H \geq \frac{1}{2}$	Cor. 2.4
Short-range dependent	Th. 2.3

fOU(1)	Def. 1.42
$U_t^{(Z,\alpha)} = \mathrm{e}^{-\alpha t} \int_{-\infty}^{t} \mathrm{e}^{\alpha s} d\widehat{Z}_s$	Def. 1.42
$\lim_{t\to\infty} \mathbf{E}\left(U_s^{(Z,\alpha)} U_{s+t}^{(Z,\alpha)}\right)$ $= \frac{1}{2}\sum_{n=1}^{N} \alpha^{-2n} \left(\prod_{k=0}^{2n-1}(2H-k)\right) t^{2H-2n} + \mathbf{O}(t^{2H-2N-2})$	Prop. 2.2
Short-range dependent, if $H < \frac{1}{2}$	Th. 2.5
Long-range dependent, if $H > \frac{1}{2}$	Th. 2.5
Y	Def. 3.1
$Y_t^{(\alpha)} = \int_0^t \mathrm{e}^{-\alpha s} dZ_{\tau_s} = \int_0^t \mathrm{e}^{-\alpha s} dZ_{\frac{H\mathrm{e}^{\frac{\alpha s}{H}}}{\alpha}}$	Def. 3.1
$\{\alpha^H Y_{\frac{t}{\alpha}}^{(\alpha)} : t \geq 0\} \stackrel{d}{=} \{Y_t^{(1)} : t \geq 0\}$	Prop. 3.3
Locally Hölder continuous of any order $\beta < H$	Prop. 3.10
$\mathbf{E}\left(Y_t^{(\alpha)} Y_s^{(\alpha)}\right) = \int_0^t (t-x) k_{\alpha,H}(x) dx + \int_0^s (s-x) k_{\alpha,H}(x) dx - \int_0^{t-s} (t-s-x) k_{\alpha,H}(x) dx$, if $H \in (\frac{1}{2}, 1)$	Prop. 3.18
fOU(2)	Def. 3.6
$U_t^{(D,\gamma)} = \mathrm{e}^{-\gamma t} \int_{-\infty}^{t} \mathrm{e}^{(\gamma-1)s} dZ_{\tau_s^{(1)}} = \mathrm{e}^{-\gamma t} \int_{-\infty}^{t} \mathrm{e}^{(\gamma-1)s} dZ_{H\mathrm{e}^{\frac{s}{H}}}$	Def. 3.6
Locally Hölder continuous of any order $\beta < H$	Prop. 3.11
$\lim_{t\to\infty} \mathbf{E}\left(U_t^{(D,\gamma)} U_s^{(D,\gamma)}\right) = C \exp^{-\min\left(\gamma, \frac{1-H}{H}\right)t}$, if $H \in (\frac{1}{2}, 1)$	Prop. 3.23
Short-range dependent, if $H \in (\frac{1}{2}, 1)$	Th. 3.24

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